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4 CONSOLIDATION THEORY

4.1 BASIC EQUATIONS OF CONSOLIDATION

The governing equations of consolidation as used in PLAXIS follow Biot's theory (Biot, 1956). Darcy's law for fluid flow and elastic behaviour of the soil skeleton are also assumed. The formulation is based on small strain theory. According to Terzaghi's principle, stresses are divided into effective stresses and pore pressures:

\[ \sigma = \sigma' + m \cdot (p_{\text{steady}} + p_{\text{excess}}) \]  

(4.1)

where:

\[ \sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})^T \]  

and:

\[ m = (1 \ 1 \ 1 \ 0 \ 0 \ 0)^T \]  

(4.2)

\( \sigma \) is the vector with total stresses, \( \sigma' \) contains the effective stresses, \( p_{\text{excess}} \) is the excess pore pressure and \( m \) is a vector containing unity terms for normal stress components and zero terms for the shear stress components. The steady state solution at the end of the consolidation process is denoted as \( p_{\text{steady}} \). Within PLAXIS \( p_{\text{steady}} \) is defined as:

\[ p_{\text{steady}} = \Sigma \cdot M_{\text{weight}} \cdot p_{\text{input}} \]  

(4.3)

where \( p_{\text{input}} \) is the pore pressure generated in the input program based phreatic lines or on a groundwater flow calculation. Note that within PLAXIS compressive stresses are considered to be negative; this applies to effective stresses as well as to pore pressures. In fact it would be more appropriate to refer to \( p_{\text{excess}} \) and \( p_{\text{steady}} \) as pore stresses, rather than pressures. However, the term pore pressure is retained, although it is positive for tension.

The constitutive equation is written in incremental form. Denoting an effective stress increment as \( \sigma' \) and a strain increment as \( \varepsilon' \), the constitutive equation is:

\[ \sigma' = M \varepsilon' \]  

(4.4)

where:

\[ \varepsilon = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})^T \]  

(4.5)

and \( M \) represents the material stiffness matrix.

4.2 FINITE ELEMENT DISCRETISATION

To apply a finite element approximation we use the standard notation:

\[ u = N \nu \quad p = \overline{N} \overline{p}_e \quad \varepsilon = B \nu \]  

(4.6)

where \( \nu \) is the nodal displacement vector, \( \overline{p}_e \) is the excess pore pressure vector, \( u \) is the continuous displacement vector within an element and \( p \) is the (excess) pore pressure. The matrix \( \overline{N} \) contains the interpolation functions and \( B \) is the strain interpolation matrix.
In general the interpolation functions for the displacements may be different from the interpolation functions for the pore pressure. In PLAXIS, however, the same functions are used for displacements and pore pressures.

Starting from the incremental equilibrium equation and applying the above finite element approximation we obtain:

\[
\int B^T d\sigma \, dV = \int N^T d\, f \, dV + \int N^T d\, t \, ds + L_0
\]  

with:

\[
L_0 = \int N^T f_0 \, dV + \int N^T L_0 \, ds - \int B^T \sigma_0 \, dV
\]

where \( f \) is a body force due to self-weight and \( t \) represents the surface tractions. In general the residual force vector, \( L_0 \), will be equal to zero, but solutions of previous load steps may have been inaccurate. By adding the residual force vector the computational procedure becomes self-correcting. The term \( dV \) indicates integration over the volume of the body considered and \( ds \) indicates a surface integral.

Dividing the total stresses into pore pressure and effective stresses and introducing the constitutive relationship gives the nodal equilibrium equation:

\[
K \, d\, V + L \, d\, p = d\, f
\]  

where \( K \) is the stiffness matrix, \( L \) is the coupling matrix and \( d\, f \) is the incremental load vector:

\[
K = \int B^T \, M \, B \, dV \quad (4.10a)
\]

\[
L = \int B^T \, m \, N \, dV \quad (4.10b)
\]

\[
d\, f = \int N^T \, d\, f \, dV + \int N^T \, d\, t \, ds \quad (4.10c)
\]

To formulate the flow problem, the continuity equation is adopted in the following form:

\[
\nabla^T \mathbf{R} \, \nabla \left( \gamma_w \, \nabla \cdot p_{steady} - p \right) + \gamma_w \, m \frac{\partial \xi}{\partial t} + n \frac{\partial p}{\partial t} = 0
\]  

where \( \mathbf{R} \) is the permeability matrix:

\[
\mathbf{R} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}
\]

\( n \) is the porosity, \( K_w \) is the bulk modulus of the pore fluid and \( \gamma_w \) is the unit weight of the pore fluid. This continuity equation includes the sign convention that \( p_{steady} \) and \( p \) are considered positive for tension.
As the steady state solution is defined by the equation:

\[ \nabla^T R \nabla \left( \gamma_w \cdot \mathbf{g} - p_{\text{steady}} \right) / \gamma_w = 0 \]  
(4.13)

the continuity equation takes the following form:

\[ \nabla^T R \nabla p / \gamma_w + m^T \frac{\partial e}{\partial t} - \frac{n}{K_w} \frac{\partial p}{\partial t} = 0 \]  
(4.14)

Applying finite element discretisation using a Galerkin procedure and incorporating prescribed boundary conditions we obtain:

\[ -H \frac{\partial p}{\partial n} + I^T \frac{d v}{d t} - S \frac{d p}{d t} = q \]  
(4.15)

where:

\[ \mathbf{H} = [\nabla \mathbf{N}^T] R \nabla \mathbf{N} / \gamma_w \ dV \right], \quad S = \int \frac{n}{K_w} N^T N \ dV \]  
(4.16)

and \( q \) is a vector due to prescribed outflow at the boundary. However within PLAXIS Version 7 it is not possible to have boundaries with non-zero prescribed outflow. The boundary is either closed or open with zero excess pore pressure. Hence \( q = 0 \). In reality the bulk modulus of water is very high and so the compressibility of water can be neglected in comparison to the compressibility of the soil skeleton. To minimize the input for consolidation analyses, a preset value for \( K_w \), is adopted which is of the form:

\[ \frac{K_w}{n} = 100 \ K_{\text{skeleton}} = \frac{100 E}{3 \ (1 - 2\nu)} \]  
(4.17)

Hence the bulk modulus of the pore fluid is taken a hundred times the bulk modulus of the soil skeleton.

The equilibrium and continuity equations may be compressed into a block matrix equation:

\[
\begin{bmatrix}
K & L \\
L^T & -S
\end{bmatrix}
\begin{bmatrix}
\frac{d v}{d t} \\
\frac{d p}{d t}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
0 & H
\end{bmatrix}
\begin{bmatrix}
v \\
p_n
\end{bmatrix}
+ 
\begin{bmatrix}
\frac{d f_n}{d t} \\
q_n
\end{bmatrix}
\]  
(4.18)

A simple step-by-step integration procedure is used to solve this equation. Using the symbol \( \Delta \) to denote finite increments, the integration gives:

\[
\begin{bmatrix}
K & L \\
L^T & -S
\end{bmatrix}
\begin{bmatrix}
\Delta v \\
\Delta p_n
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
0 & H
\end{bmatrix}
\begin{bmatrix}
\n0 \\
0
\end{bmatrix}
\begin{bmatrix}
\n0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta f_n \\
\Delta q_n
\end{bmatrix}
\]  
(4.19)
where:
$$S^* = \alpha \Delta t \left( S + \frac{t}{H} \right)$$
$$q_n^* = q_{n0}^0 + \alpha \Delta q_n$$  \hspace{1cm} (4.20)

and \(y_n^0\) and \(p_{n0}\) denote values at the beginning of a time step. The parameter \(\alpha\) is the time integration coefficient. In general the integration coefficient \(\alpha\) can take values from 0 to 1. In PLAXIS the fully implicit scheme of integration is used with \(\alpha = 1\).

**4.3 ELASTOPLASTIC CONSOLIDATION**

In general, when a non-linear material model is used, iterations are needed to arrive at the correct solution. Due to plasticity or stress-dependent stiffness behaviour the equilibrium equations are not necessarily satisfied using the technique described above. Therefore the equilibrium equation is inspected here. Instead of Eq. (4.9) the equilibrium equation is written in sub-incremental form:

$$r = p + v K$$  \hspace{1cm} (4.21)

where \(r_n\) is the global residual force vector. The total displacement increment \(\Delta v\) is the summation of sub-increments \(\delta v\) from all iterations in the current step:

$$r_n = \int N^T \delta V + \int N^T I d s \cdot \int B^T \sigma d V$$  \hspace{1cm} (4.22)

with:

$$\delta V = \int f_0 + \Delta f$$  \hspace{1cm} (4.23)

In the first iteration we consider \(\sigma = \sigma_0\), i.e. the stress at the beginning of the step. Successive iterations are used on the current stresses that are computed from the appropriate constitutive model.