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2 ELASTICITY PROBLEMS WITH KNOWN THEORETICAL SOLUTIONS

A series of elastic benchmark calculations is described in this Chapter. In each case the analytical solutions may be found in many of the various textbooks on elasticity solutions, for example Giroud (1972) and Poulos & Davis (1974).

2.1 SMOOTH RIGID STRIP FOOTING ON ELASTIC SOIL

**Input:** The problem of a smooth strip footing on an elastic soil layer with depth $H$ is shown in Fig. 2.1. This figure also shows relevant soil data and the finite element mesh used in the calculation. A uniform vertical displacement of 10 mm is prescribed to the footing and the indentation force, $F$, is calculated from the results of the finite element calculation. Since the problem is symmetric it is possible to model only one half of the situation as shown in Fig. 2.1.

![Figure 2.1 Problem geometry](image)

**Output:** The scaled deformation of the finite element mesh at the end of the elastic analysis is shown in Fig. 2.2. The footing force resulting from a rigid indentation of 10 mm is calculated to be $F = 15.26$ kN. (Note that when only half of the elastic halfspace is modelled the force calculated by Plaxis will be exactly one half of this value).

\[
\text{Settlement} = \left( \frac{F \delta}{2 (1 + \nu) G} \right) \quad \text{with} \quad \delta = 0.88 \quad \text{for} \quad \frac{H}{\frac{1}{2}B} = 4
\]

**Verification:** Giroud (1972) gives the analytical solution to this problem in the formula above, where $H$ is the depth of the layer, $B$ is the total width of the footing and $\delta$ is a constant. For the dimensions and material properties used in the finite element analysis this solution gives a footing force of 15.15 kN. The error in the numerical solution is therefore about 0.7%.
Fig. 2.3 gives both the analytical and numerical results for the pressure distribution underneath the footing. This figure shows that the numerical results agree very well with the analytical solution.

Figure 2.2 Deformed mesh

Figure 2.3 Pressure distribution at footing
2.2 STRIP LOAD ON ELASTIC GIBSON SOIL

Input: Fig. 2.4 shows the mesh and the soil data for a plane strain calculation of the settlement of a strip load on Gibson soil. (Gibson soil is an elastic layer in which the shear modulus increases linearly with depth). Using \( z \) to denote depth, the shear modulus, \( G \), used in the calculation is given by: \( G = \alpha z = 100z \). With a Poisson's ratio of 0.495, the Young's modulus varies by: \( E = 299z \). In order to prescribe this variation of Young's modulus in the material properties window the reference value of Young's modulus, \( E_{ref} \), is taken very small and the Advanced option is selected from the Parameters tab sheet. The increase of Young's modulus \( E_{increment} \) is set to 299 and the reference level \( y_{ref} \) is entered as 4.0 m, being the top of the geometry.

Output: An exact solution to this problem is only available for the case of a Poisson's ratio of 0.5; in the PLAXIS calculation a value of 0.495 is used for the Poisson's ratio in order to approximate this incompressibility condition. The numerical results show an almost uniform settlement of the soil surface underneath the strip load as can be seen from the velocity contour plot in Fig. 2.5. The computed settlement is 0.047 m at the centre of the strip load.
Verification: The analytic solution is exact only for an infinite half-space, whereas the PLAXIS solution is obtained for a layer of finite depth. However, the effect of a shear modulus that increases linearly with depth is to localise the deformations near the surface; it would therefore be expected that the finite thickness of the layer will only have a small effect on the results. The exact solution for this particular problem, as given by Gibson (1967), gives a uniform settlement beneath the load of magnitude:

\[
\text{Settlement} = \frac{p}{2 \alpha}
\]

In this case the exact solution gives a settlement of 0.05 m. The numerical solution is 6% lower than the exact solution.
2.3 BENDING OF BEAMS

Input: For the verification of beams two problems are considered. These problems involve a single point load and a uniformly distributed load on a beam respectively, as indicated in Fig. 2.7. For these problems the characteristics of a HEB 200 steel beam have been adopted. In a plane strain model, the beam is in fact a plate of 1 m width in the out of plane direction. The properties, the dimensions and the load of the beam are:

\[ EA = 1.64 \times 10^6 \text{ kN} \quad EI = 1200 \text{ kNm}^2 \quad \nu = 0.0 \]

\[ l = 2 \text{ m} \quad F = 100 \text{ kN} \quad q = 100 \text{ kN/m} \]

Beams cannot be used individually. A single block cluster may be used to create the geometry. The two beams are added to the bottom line with a spacing in between. Use point fixities on the end points of the beam. A very coarse mesh is sufficient to model the situation. In the Initial conditions mode the soil cluster can be deactivated so that only the beams remain.

Output: The results of the two calculations are plotted in Figs 2.8, 2.9 and 2.10. For the extreme moments and displacements we find:

- **Point load:** \( M_{\text{max}} = 50.0 \text{ kNm} \quad u_{\text{max}} = 14.0 \text{ mm} \)
- **Distributed load:** \( M_{\text{max}} = 50.0 \text{ kNm} \quad u_{\text{max}} = 17.4 \text{ mm} \)
Figure 2.9 Shear forces

Figure 12.10 Computed displacements

**Verification:** As a first verification, it is observed from Figs 2.8a and 2.8b that PLAXIS yields the correct distribution of moments. For further verification we consider the well-know formulas as listed below. These formulas give approximately the values as obtained from the PLAXIS analysis.

Point load:  
\[ M_{\text{max}} = \frac{1}{4} Fl = 50 \text{ kNm} \quad \text{and} \quad u_{\text{max}} \approx \frac{1}{48} \frac{Fl^3}{EI} = 13.9 \text{ mm} \]

Distributed load:  
\[ M_{\text{max}} = \frac{1}{8} ql^2 = 50 \text{ kNm} \quad \text{and} \quad u_{\text{max}} \approx \frac{5}{384} \frac{ql^4}{EI} = 17.36 \text{ mm} \]
2.4 BENDING OF PLATES

**Input:** In an axisymmetric analysis, beams may be used as circular plates. The latter two verification examples involve a uniformly distributed load \( p \) on a circular plate. In one example the plate can rotate freely at the boundary and in the other example the plate is clamped, as indicated in Fig. 2.11.

![Figure 2.11 Loading scheme for testing axisymmetric plates](image)

**Solution:** For the situation of a circular plate with a uniformly distributed load one can elaborate and solve a differential equation. The analytical solutions for this equation depend on the boundary conditions. For the plate with free rotation at the boundary one finds:

**Settlement:**

\[
 w = \frac{p \ R^4}{64 \ D} \left( \frac{5+\nu}{1+\nu} - \frac{6+2\nu}{1+\nu} \frac{r^2}{R^2} + \frac{r^4}{R^4} \right) \quad D = \frac{EI_{plaxis}}{1-\nu^2}
\]

**Moments:**

\[
 m_{rr} = \frac{p \ R^2}{16} \left( (3+\nu) - (3+\nu) \frac{r^2}{R^2} \right) \quad \quad \quad m_{tt} = \frac{p \ R^2}{16} \left( (3+\nu) - (1+3\nu) \frac{r^2}{R^2} \right)
\]

Using \( R = 1 \text{ m}, \ d = 0.1 \text{ m}, \ p = 1 \text{ kN/m}^2, \ \nu = 0 \) and \( EI = 1 \text{ kNm}^2/\text{m} \), this gives:

In the centre: \( w = 0.078125 \text{ m} \) Numerical: \( w = 0.07862 \text{ m} \)

\( m_{rr} = 0.18750 \text{ kNm/m} \) Numerical: \( m_{rr} = 0.18797 \text{ kNm/m} \)

At \( r = R/2 \): \( w = 0.055664 \text{ m} \) Numerical: \( w = 0.056039 \text{ m} \)

\( m_{rr} = 0.140625 \text{ kNm/m} \) Numerical: \( m_{rr} = 0.140601 \text{ kNm/m} \)

For the plate with a clamped boundary one finds:

**Settlement:**

\[
 w = \frac{p \ R^4}{64 \ D} \left( 1 - \frac{r^2}{R^2} \right)^2
\]
Moments:

\[ m_{rr} = \frac{P}{16} \frac{R^2}{(1 + \nu) - (3 + 3\nu) \frac{L^2}{R^2}} \]

\[ m_{\theta} = \frac{P}{16} \frac{R^2}{(1 + \nu) - (1 + 3\nu) \frac{L^2}{R^2}} \]

Using \( R = 1 \text{ m}, d = 0.1 \text{ m}, \rho = 1 \text{ kN/m}^2, \nu = 0 \) and \( EI = 1 \text{ kNm}^2/\text{m} \), this gives:

In the centre: \( w = 0.01563 \text{ m} \) Numerical: \( w = 0.01613 \text{ m} \)

\( m_{rr} = 0.06250 \text{ kNm/m} \) Numerical: \( m_{rr} = 0.06274 \text{ kNm/m} \)

At \( r = R/2 \): \( w = 0.008789 \text{ m} \) Numerical: \( w = 0.009164 \text{ m} \)

\( m_{rr} = 0.015625 \text{ kNm/m} \) Numerical: \( m_{rr} = 0.015622 \text{ kNm/m} \)

At \( r = R \): \( m_{rr} = -0.125 \text{ kNm/m} \) Numerical: \( m_{rr} = -0.125 \text{ kNm/m} \)

**Verification:** The settlement difference is mainly due to shear deformation, which is included in the numerical solution but not in the analytical solution. Apart from this, the numerical results are very close to the analytical solution.
2.5 PERFORMANCE OF SHELL ELEMENTS

A beam in PLAXIS can be applied as a tunnel lining. By using this element, 3 types of deformations are taken into account: shear deformation, compression due to normal forces and obviously bending.

**Input:** A ring with a radius of \( R = 5 \) m is considered. The Young's modulus and the Poisson's ratio of the material are taken respectively as \( E = 10^6 \) kPa and \( \nu = 0 \). For the thickness of the ring cross section, \( H \), several different values are taken so that we have rings ranging from very thin to very thick. In order to model such a ring the bottom point of the ring is fixed with respect to translation and the top point is allowed to move only in the vertical direction. Then the load \( F = 0.2 \) kN/m is applied only at the top point. Geometric non-linearity is not taken into account.

**Output:** The calculated vertical deflections at the top point are presented in Fig. 2.14. The deformed shape of the ring is also shown in Fig. 2.14. The calculated normal force at the belly of the ring is 0.50 for all different values of ring thickness. The calculated bending moment at the belly varying from 0.182 to 0.189 as the ring changes from thin to thick. Typical graphs of the bending moment and normal force are shown in Figs. 2.15a and 2.15b.

![Figure 2.14 Calculated deflections compared with analytical solutions](image-url)
Figure 2.15a Normal forces  Figure 2.15b Bending moments

Verification: The analytical solution for the deflection of the ring is given by Blake (1959), and the analytical solution for the bending moment and the normal force can be found from Roark (1965). The vertical displacement at the top of the ring is given by the following formula:

$$\delta = \frac{F \lambda}{E} \left[ 1.788 \lambda^2 + 3.091 \frac{0.637}{1+12 \lambda^2} \right] \text{ with } \lambda = \frac{R}{H}$$

The solid curve in Fig. 2.14 is plotted according to this formula. It can be seen that the deflections calculated by PLAXIS fit the theoretical solutions very well. Only for a very thick ring some errors are observed, which is about 7 per cent for $H/R=0.5$. But for thin rings the error is nearly zero. The analytical solution for the bending moment and normal force at the belly is 0.182 and 0.5 respectively. Thus even for very thick rings the error in the bending moment is just 4 per cent, and the error in the normal force is only 0.2 percent.
2.6 UPDATED MESH ANALYSIS OF A CANTILEVER

The range of problems with known solutions involving large displacement effects that may be used to test the large displacement options in PLAXIS is very limited. The large displacement elastic bending of a cantilever beam, however, is one problem which is well suited as a large displacement benchmark problem since a known analytical solution exists, Mattiasson (1981).

Geometry non-linearity is of major importance in problems involving slender structural members like beams, plates and shells. Indeed, phenomena like buckling and bulging cannot be described without considering geometry changes. Soil bodies, however, are far from slender and consequently, most finite element formulations tacitly disregard changes in geometry. This also applies to conventional PLAXIS calculations. Users should check such results by considering the truly deformed mesh. In most practical cases this will indicate very little change of geometry. In some particular cases, however, it may be significant.

For special problems of extreme large deformation an Updated mesh analysis is needed. For this reason PLAXIS involves a special module. For details on the implementation the reader is referred the PhD thesis by Van Langen (1991). This module was programmed using the Updated Lagrangian formulation as described by McMeeking and Rice (1975).

Analysis: The analysis relates to the calculation of the horizontal and vertical tip displacement for the cantilever beam shown in Fig. 2.16. The mesh used in the PLAXIS analysis is shown in Fig. 2.17. (Note that it is necessary to use a finite depth of beam in the numerical calculation in contrast to the analytical solution which is based on a beam of zero geometric thickness).

Figure 2.16 True deformation of elastic cantilever
Results: The computed load-displacement curves are plotted in Fig. 2.18. The numerical results are clearly in close agreement with the analytical solution.