Chapter 2: Introduction to Dynamics

Basic Quantities from Earthquake Records

Earthquakes are complex, dynamic events. In order to describe and work with them, a vocabulary of terms is necessary. Terms describing the intensity or magnitude of an earthquake give the engineer a quick estimate of the degree of shaking and damage one would expect. Other values reflect the time history of the earthquake such as acceleration, velocity, displacement, power, and other values derived from the time histories.

The size of an earthquake has been reported in several ways. Historically the earthquake intensity was a qualitative description of the earthquake’s ability to cause damage. The Rossi-Forel (RF) scale describing intensities with values ranging from I to X was developed in 1880’s and used for many years. The modified Mercalli intensity (MMI) scale is a better way to represent conditions in the U.S. and was adopted there in the 1930’s. The MMI scale is based on performance of familiar structures and is shown in Table 2-1.

Include discussion on Richter, Surface wave, body wave, as well as energy, and Moment magnitude.

Shown below is a record of earthquake acceleration versus time. Note the irregular shape of the record. However, one can differentiate portions of strong shaking from portions of weaker shaking. One can also identify, approximately, the frequency of strong motion in the record. The maximum acceleration during the event is about 0.32g or 3.1 m/sec² at time = 17 seconds.

![Acceleration vs. Time](image)

Figure 1. Acceleration vs. Time Record (El Centro, California, 1940)

The record starts with a relatively calm period, mainly due to the arrival of p-waves to the accelerometer. These waves travel faster than the more-damaging s-waves and r-waves.

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and triggered the seismograph. Strong motion starts at about 16 seconds and lasts until about 42 seconds, depending on one’s definition of strong motion. Since this is a digital record, we can study any portion of the record more carefully and calculate several other quantities of interest to the earthquake engineer. This particular record contains 4187 points, recorded at time intervals of 0.02 seconds. These records are readily available for downloading at the United States Geological Survey web site (usgs.gov).

This record has been corrected and adjusted from its original form. One can determine this by integrating the time series to produce a velocity record that begins and ends at 0.0; then integrating the velocity to generate a displacement record that begins at 0.0 and ends at the final coordinates measured after the event (usually 0.0 as well, but there could be permanent displacements). An uncorrected record would show “drift” where the velocity and displacements would not return to zero. Drift is due mainly to the electronic and mechanical imperfections in the recording device. There are several standard methods for correcting acceleration records, but we will leave it to the seismologist.

Instead, let’s look more closely at the acceleration record between 16 and 20 seconds as shown in figure 2. The acceleration is indeed irregular, but there are some predominant frequencies one can make out. Between 16.70 and 17.20 seconds is a large wave; the corresponding frequency f = 2.0 Hz. or ω = 6.28 radians/sec. One could discover more waves within the record; however, there are better methods for doing this discussed later.

Figure 2. Acceleration vs. Time at t = 16.00 to 20.00 seconds

Note also in figure 2 the actual data points, this is probably a hand-digitized record of an analog record. More modern equipment can record at faster rates, typically 500-1000 points every second. Beyond this speed, it becomes a pointless exercise because there is little useful information to be gained at higher sampling rates; the earthquake motion does not contain large quantities of very high-frequency signal.
In order to better quantify the earthquake and a structure’s response to earthquakes some simplifying methods are used. Most important in those methods is the concept of harmonic motion, that is, motion described by

\[ x = A \sin(\omega t - \varphi) \quad \dot{x} = -\omega A \cos(\omega t - \varphi) \quad \ddot{x} = -\omega^2 A \sin(\omega t - \varphi) \]

where \( x \) = displacement, \( \dot{x} \) = velocity, \( \ddot{x} \) = acceleration

\( A = \text{amplitude of wave} \)

\( \omega = \text{frequency (radians/sec)} \)

\( t = \text{time} \)

\( \varphi = \text{phase lag (radians)} \)

The phase lag accounts for shifting along the time axis. Typical harmonic motion is shown in figure 3.

**Figure 3. Single degree of freedom system with initial displacement, no driving function**

Amplitude is sometimes called “single amplitude” since it measures the distance from zero to maximum (or minimum). “Double amplitude” would mean from maximum to minimum values. Frequency, \( f \), and circular frequency, \( \omega \), are related by \( 2\pi f = \omega \). Earthquakes can be represented as a sum of harmonic motions with different frequencies, amplitudes, and phases. The frequencies are related to each other as specific multiples, amplitudes and phases and are calculated via Fourier Transform. This process is very useful in determining the “frequency content” of an earthquake, that is, how much low frequency component versus high frequency component an earthquake contains.

**Fourier Transform, Frequency Domain**

The Fourier transform is calculated on a discreet time series of acceleration, velocity, or displacement, such as the earthquake record shown in figure 1. Since the time series and frequency spectrum are both discreet, the transform is designated DFT or discreet Fourier transform. Computation of the DFT follows:
Given a digitized record of acceleration $\ddot{x}(t)$ with $N$ points at a time interval $\Delta t$, the series can be decomposed into $N/2 + 1$ harmonics as follows:

$$\ddot{x}(t) = \text{Re} \sum_{s=0}^{N/2} \tilde{X}_s e^{i\omega_s t}$$  \hspace{1cm} \text{Equation 2-2}

where

$$\omega_s = \frac{2\pi s}{N \Delta t}, \quad s = 0, 1, 2, \ldots, \frac{N}{2}$$ \hspace{1cm} \text{Equation 2-3}

and the $\tilde{X}_s$ coefficients are the complex-valued Fourier amplitudes

$$\tilde{X}_s = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \ddot{x}_k e^{-i\omega_s k \Delta t}, & \text{for } s = 0, s = \frac{N}{2} \\ \frac{2}{N} \sum_{k=0}^{N-1} \ddot{x}_k e^{-i\omega_s k \Delta t}, & \text{for } 1 \leq s \leq \frac{N}{2} \end{cases}$$ \hspace{1cm} \text{Equation 2-4}

Note that the coefficients are one half the magnitudes at the beginning and end of the series. The counter, $k$ starts at zero, implying the first coefficient has a frequency $\omega = 0.0$ associated with it. In this formula, the time series is regularly spaced according to the formula

$$x_k = \ddot{x}(k \Delta t), \quad k = 0, 1, 2, \ldots, N - 1$$ \hspace{1cm} \text{Equation 2-5}

Euler’s identity allows us to compute equation 2-4 as trigonometric functions

$$e^{-i\omega_s k \Delta t} = \cos(\omega_s k \Delta t) - i \sin(\omega_s k \Delta t)$$ \hspace{1cm} \text{Equation 2-6}

From a computer programming aspect, the calculation of coefficients takes place within two loops; the outer loop increments the value for $s$ and therefore the frequency term. The inner, faster loop performs the summation over $k$-values of time series and in most cases the summation of the real and imaginary parts are kept as separate terms, to be combined later.

The computation of the complex amplitudes $\tilde{X}_s$ from the given real values $\ddot{x}_k$ is most conveniently made by a fast algorithm known as the “Fast Fourier Transform” (FFT) by Cooley and Tukey (1965). This algorithm produces the results of equation 2-4 in a time proportional to $N \log(N)$ as opposed to $N^2$ by the obvious method. The same algorithm can produce the inverse transformation from complex amplitudes $\tilde{X}_s$ (the frequency domain) to the real values $\ddot{x}_k$ (the time domain). A limitation on the use of the fast Fourier transform method is that $N$ must be a power of 2, (radix 2). This restriction is not important since it is always possible, and in fact desirable to augment the earthquake by a string of trailing zeros. The latter is so because the motion given by equation 2-4 is periodic with the period

$$T = N \Delta t$$ \hspace{1cm} \text{Equation 2-7}

Hence, in order to simulate the finite duration of actual earthquakes it is necessary to introduce a “quiet zone” at the end of each cycle to allow the viscous damping of the system time to attenuate the response from one cycle before the beginning of the next cycle.

Since the value of $\tilde{X}_s$ is complex, it is often written as a magnitude and phase instead of real and imaginary components. These terms are interchangeable with the relationship

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\[ \text{Mag } \dot{X}_S = \sqrt{\Re \dot{X}_S^2 + \Im \dot{X}_S^2} \quad \text{Equation 2-8} \]

and

\[ \varphi = \tan^{-1}\left( \frac{\Im \dot{X}_S}{\Re \dot{X}_S} \right) \quad \text{Equation 2-9} \]

The magnitude is a measure of the content of a given frequency in the earthquake. The phase is a measure of how the harmonic is positioned along the time axis with respect to the other harmonics. Fourier spectra are not only useful in earthquake studies, but many other problems in soil dynamics (machine foundations, railroad, bridge dynamics, and field measurements) and should become a part of any geotechnical engineer’s “toolkit”. Given the El Centro acceleration record of figure 1, the Fourier Transform is shown below. The record has been trimmed to 4096 values (in keeping with requirements of FFT to be \(2^n\) values). It would have been possible to perform the analysis on 8192 values, but that seems excessive for this exercise.

![Fourier Transform of El Centro Acceleration Record](image)

**Figure 4** Magnitudes of Fourier Transform; El Centro Acceleration Record

Note that the predominant frequencies are in the range from \(\omega=7\) to 14 \(\text{r/sec}\). The higher frequencies tend to die out simply because there isn’t much shaking at those frequencies. The plot of phase angle would not be very helpful since its values range between \(\pi/2\) and \(-\pi/2\) and there is little pattern to discern any information. This is not true, however for other field applications where phase is very important (eg. SASW method for determining wave velocities at a site). Appendix C lists typical computer codes for computation of DFT and FFT. Also packaged with this module is an Excel spreadsheet for computing DFT.

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The use of harmonics is very important in studying earthquake response of structures. It forms the basis for many other approaches, including EuroCode methods. There are some other related calculations necessary for analysis. Elastic and plastic response spectra are two such computations. They are not the same as FFT, but they are similar. They are discussed in the next sections.

**SDOF systems, Response Spectra**

One way to evaluate response of a structure to earthquake motion is to model it as a single-degree-of-freedom system. This is the most fundamental sort of model one can create for a structure. Nonetheless, it is very useful to the engineer to evaluate the dynamic response of a structure to a given earthquake. Before proceeding with earthquake response analysis, it is necessary to review some concepts of basic vibration and single degree of freedom systems (SDOF). A SDOF system consists of mass, dashpot, spring, and some driving function.

\[
m \dddot{x} + c \ddot{x} + kx = P_0 \sin(\omega t)
\]

where  
\( m = \) system mass  
\( c = \) system damping (dashpot)  
\( k = \) system stiffness (spring constant)  
\( x = \) displacement  
\( \dot{x} = \) velocity  
\( \ddot{x} = \) acceleration  
\( P_0 \sin(\omega t) = \) driving function, typically a machine

If there is no driving function, the system is a “free vibrating” system and is driven only by initial conditions of displacement, or velocity, or both. This is analogous to a pendulum initially displaced and swinging back and forth. The SDOF equation (2-10) is often re-cast with the value of damping expressed as a damping ratio (D) where \( D = \frac{c}{c_{crit}} \) and \( c_{crit} = \sqrt{km} \). Typically, damping is much less than critical, perhaps 2%-5% (D=0.02-0.05) of that value. Most structures are assumed to have this much damping. Foundations may have more, perhaps 10-25% damping due to the way foundations dissipate energy. Recall that, if the structure has 0% damping it will shake forever if excited by an earthquake or machine. A more complete discussion of SDOF systems is given in Appendix A.

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Earthquake excitation (figure 5b) is nearly the same. There is no forcing function on the right-hand side as in equation 2-10, but the inertia force, generated by mass and acceleration is different.

\[ m\dddot{x} + c\dddot{x} + k\ddot{x} = 0 \quad \text{Equation 2-11} \]

where \( x_t \) represents the total displacement of the mass with respect to some reference axis. This displacement is different from the relative displacement \( x \), seen in the damping and spring terms. One can rewrite equation 2-11 by splitting \( x_t \) into two components \( x \) and \( x_g \) as seen in figure 4b;

\[ m\dddot{x} + m\dddot{x}_g + c\dddot{x} + k\ddot{x} = 0 \quad \text{or} \quad m\dddot{x} + c\dddot{x} + k\ddot{x} = -m\dddot{x}_g = P_{\text{earthquake}}(t) \quad \text{Equation 2-12} \]

The negative sign has little meaning since the earthquake will move in both directions. The important point is that earthquake forces are generated by the inertial resistance of the structure. Recall what was said about light structures and structures with less mass near the top. Since earthquake forces in the building are due to the building’s own mass, less mass translates to less force.

What is more interesting to the earthquake engineer is how their structure might respond to a given earthquake. This is evaluated using a method to determine response of a SDOF system to a general excitation history, such as an earthquake. Duhamel’s integral is the accepted method for determining displacement response of a SDOF system to an arbitrary loading history. A full discussion of Duhamel’s integral is presented in Appendix B. An elastic response spectrum is a summary of the maximum displacements of different SDOF structures to a given earthquake. The structures analyzed have different natural frequencies of shaking determined by

\[ \omega_s = \sqrt{\frac{k}{m}} \quad \text{undamped systems}; \quad \omega_d = \sqrt{\frac{k}{m}(1 - D^2)} \quad \text{damped systems} \quad \text{Equation 2-13} \]

This can be easily seen by setting mass, \( m=1.0 \), damping ratio \( D=0.0 \), and \( k = \) progressively larger values to generate higher natural frequencies. A typical response spectrum uses damping ratios of 0.02 (2%) and 0.05 (5%) as well, to illustrate the effects of structural damping on response. Remember that an earthquake response spectrum is unique to the earthquake, not the structure. A displacement response spectrum of the earthquake of figure 1 is shown below (figure 6). Each point on the spectrum represents the maximum displacement experienced by a SDOF structure with a specified natural frequency and damping ratio, subjected to the El Centro Earthquake. For example, a structure with a natural frequency of 2.4 rad/sec and 2% damping would experience a maximum horizontal displacement of 4.0 centimeters. This is highlighted on figure 6. Similar spectra can be produced for velocity and acceleration, either by differentiating the displacement response history or simply scaling maximum displacements by a factor of \( \omega \) for velocity and \( \omega^2 \) for acceleration (note the relationship between amplitudes in equation 2-1). When scaled by the factor \( \omega \), the spectrum is called pseudo-acceleration, pseudo-velocity or pseudo-displacement.

Response spectra are used in design to represent an envelope of maximum expected displacements, velocities, and accelerations for “design” levels of shaking. If a recorded acceleration record, such as El Centro, is used for design in Győr, it must be scaled back such that the maximum levels of shaking fit within the design spectrum limits for Eurocode design. Methods to do this, and to use design spectra directly, are discussed in later sections.
Inelastic spectra incorporate an additional property, ductility, into the response equation. Suppose that the structure in figure 4a,b could also form a plastic hinge in its columns if deflected far enough. This would drastically reduce the overall stiffness of the structure, allowing it to absorb more energy (beyond the damping component). Computing an inelastic response spectrum requires a numerical model with a greater degree of sophistication than the previous one. Inelastic spectra are discussed in greater detail in Appendix D.

Figure 6. Displacement Response Spectrum

**Response of Multi-Degree of Freedom Systems**

One may write separate equations for multi-degree of freedom systems similar to equation 2-10 for more than one mass, \((m_i)\), and more than one component of motion \((x_i)\). The component of motion could describe a different displacement direction of a point of interest (node), or could describe the motion of a different node. Similarly, the driving function is specific to that node or direction, or for earthquake loading, the inertial component is divided as in equation 12. Examples of multi-degree-of-freedom (MDOF) are shown in figure 7a,b. The structure in 7a is a stiff-floor, flexible column model as in figure 5. It is allowed to move in only the horizontal direction and therefore has 3 degrees of freedom \((x_1, x_2, \text{and } x_3)\). This structure will be useful later in the discussion on modal analysis. The beam in 7b may be a continuous beam and may be analyzed with any number of nodes \((\geq 3)\). The number and location of nodes is left to the analyst. He should pick locations of interest (changes in properties of beam, or where other masses are attached) and at regular intervals to simplify his own interpretation of results. The number of nodes does not have to be large in order to obtain a sufficiently accurate model. The degrees of freedom may be vertical as shown \((y_1...y_5)\) or vertical and rotational \((\theta_1...\theta_5)\). If both are used, the beam would have 10 degrees
of freedom and require 10 equations of motion to solve. It should be obvious that one must resort to matrix methods very soon or be overwhelmed with equations, nodes, and properties.

Formulation of the equations of motion requires some knowledge about how forces interact in a (somewhat) complex structure. First, the structure is assumed to behave linearly, so the principle of superposition applies. Second, the stiffness, damping and mass values for 7a are known, for 7b they must be calculated. Both 7a and 7b boundary conditions can be computed in the same way. If the student is familiar with matrix structural analysis or finite elements, this process is very similar. If the student is not familiar with these methods, now is a good time to learn.

In general, four types of forces will be involved at any node: the externally applied load $p(t)$, and the forces resulting from motion: inertia $f_i$, damping $f_c$, and spring $f_s$. Thus, for each of the nodes’ degrees of freedom, dynamic equilibrium may be expressed as:

$$f_{11} + f_{D1} + f_{S1} = p_1(t)$$
$$f_{12} + f_{D2} + f_{S2} = p_2(t)$$
$$f_{13} + f_{D3} + f_{S3} = p_3(t)$$

or when the force vectors are represented in matrix form,

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} f_D \\ f_D \\ f_D \end{bmatrix} + \begin{bmatrix} f_S \\ f_S \\ f_S \end{bmatrix} = \begin{bmatrix} p(t) \\ p(t) \\ p(t) \end{bmatrix}$$

which is the MDOF equivalent of equation 2-10. Each of the resisting forces is expressed by means of a set of influence coefficients (spring, damping, mass are no longer sufficient to describe the structural system since the arrangement of beams, columns etc., has an influence as well as modulus and moment of inertia of the structural elements). Consider the elastic

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force component developed at point 2 in 7b. It will depend on all the displacements throughout the structure. Assume for now that we are dealing only with displacements (5 DOF). The elastic component may be written as:

\[ f_{s2} = k_{21}y_1 + k_{22}y_2 + k_{23}y_3 + k_{24}y_4 + k_{25}y_5 \quad \text{for more dof} \quad \cdots + k_{2N}y_N \quad \text{Equation 2-16} \]

for \( y_3 \) and \( y_4 \) we could write similar equations, more generally,

\[ f_{si} = k_{i1}y_1 + k_{i2}y_2 + k_{i3}y_3 + k_{i4}y_4 + k_{i5}y_5 \quad \text{for more dof} \quad \cdots + k_{iN}y_N \quad \text{Equation 2-17} \]

The coefficients, \( k_{ij} \) are called stiffness influence coefficients defined as follows:

\[ k_{ij} = \text{force corresponding to coordinate } i \quad \text{due to unit displacement of coordinate } j \quad \text{Equation 2-18} \]

In matrix form, the complete set of relationships may be written

\[
\begin{bmatrix}
  f_{s1} \\
  f_{s2} \\
  \vdots \\
  f_{si}
\end{bmatrix} =
\begin{bmatrix}
  k_{11} & k_{12} & \cdots & k_{1N} \\
  k_{21} & k_{22} & \cdots & k_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  k_{i1} & k_{i2} & \cdots & k_{iN}
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_i
\end{bmatrix}
\]

\text{Equation 2-19}

in which the matrix of coefficients \( k_{ij} \) is call the stiffness matrix of the structure (for the specified set of displacement coordinates) and \( \{y\} \) is the displacement vector representing the displaced shape of the structure. By analogy with equation 19, damping forces and inertia forces can be represented by similar matrices of damping influence coefficients and mass influence coefficients respectively. In each matrix, the coefficients are defined by:

\[ c_{ij} = \text{force corresponding to coordinate } i \quad \text{due to unit velocity of coordinate } j \quad \text{Equation 2-20} \]

\[ m_{ij} = \text{force corresponding to coordinate } i \quad \text{due to unit acceleration of coordinate } j \quad \text{Equation 2-21} \]

Substituting these matrix expressions into the force equilibrium equation 2-15 yields, in matrix form

\[ m\ddot{x} + c\dot{x} + kx = p(t) \quad \text{Equation 2-22} \]

where each degree of freedom has an equation of motion. It only remains to compute appropriate values of mass, damping, and stiffness. For the example in figure 7a, we can displace the first degree of freedom, \( x_1 \) one unit, hold the other nodes in place and compute the forces necessary. and compute the


**Solution of Undamped System by Modal Analysis**

The equation of motion for a MDOF-undamped system is identical to equation 2-22 with the damping matrix and velocity terms removed:

\[ m\ddot{x} + kx = 0 \quad \text{Equation 2-23} \]
where \( \mathbf{0} \) is a vector. If one assumes harmonic motion of the system, then for all degrees of freedom,

\[
x(t) = \hat{x} \sin(\omega t - \varphi)
\]  

\textbf{Equation 2-24}

in this equation, \( \hat{x} \) represents the shape of the system (does not change with time, only the amplitude varies with time) and \( \varphi \) is the phase angle. Taking the second derivative of equation 2-24 with respect to time yields acceleration as

\[
\ddot{x}(t) = -\omega^2 \hat{x} \sin(\omega t - \varphi) = -\omega^2 x
\]  

\textbf{Equation 2-25}

Substituting equation 2-24 and 2-25 into 2-23 gives

\[
-\omega^2 \hat{x} \sin(\omega t - \varphi) + k \hat{x} \sin(\omega t - \varphi) = 0
\]  

\textbf{Equation 2-26}

which can be written by dividing through by the \( \sin() \) term

\[
[k - \omega^2 m] \hat{x} = 0
\]  

\textbf{Equation 2-27}

Equation 2-27 represents the frequency equation of the system. Mathematically, this is an eigenvalue problem, with \( \omega^2 \) representing \( N \) eigenvalues and \( \hat{x} \) representing the \( N \) eigenvectors. The problem can be solved with many modern numerical packages, or by hand for the first few (lowest) frequencies via Stodola or Holzer methods.

**Example 1.**

Given the structural system shown in figure 7a, with mass and stiffness values shown, determine fundamental frequencies: \( \omega_1, \omega_2, \omega_3 \) and mode shape vectors \( \hat{x}_1, \hat{x}_2, \hat{x}_3 \) for the three degrees of freedom. ***insert example problem here where frequency and mode shapes are computed as in CP 12-1, 12-2***

**Dynamic Analysis by Modal Methods**

Finally, we are ready to put all this knowledge to use. We want to analyze a system that will tell us directly the forces transferred to a foundation and structure by an earthquake. Modal analysis makes use of the concepts discussed earlier.

In the preceding discussion of any arbitrary \( N \)-DOF system, the displaced position was defined by the \( N \) components of the displacement vector \( x \). However, for dynamic response analysis of linear systems, a much more useful representation of the displacements is provided by the free-vibration mode shapes. These shapes constitute \( N \) independent displacement patterns, the amplitudes of which may serve as generalized coordinates to express any form of displacement. The mode shapes serve the same purpose as the trigonometric functions in a Fourier series, and they are advantageous for the same reasons: because of their orthogonality properties and because they describe the displacements efficiently so that good approximations can be made with few terms.

Consider, for example, the cantilever column shown in figure 2-83-1, for which the deflected shape is defined by translational displacement coordinates at three levels.

Any displacement vector \( x \) for this structure can be developed by superposing suitable amplitudes of the three modes of vibration, as shown. For any modal component \( \hat{x}_n \) the displacements are given by the mode-shape vector \( \varphi_n \) multiplied by the modal amplitude \( X_n \), thus

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The total displacement is then obtained as the sum of the modal components,

\[ x = \varphi_1 X_1 + \varphi_2 X_2 + \ldots + \varphi_N X_N = \sum_{n=1}^{N} \varphi_n X_n \]

Equation 2-29

or, in matrix notation,

\[ \mathbf{x} = \mathbf{\Phi} \mathbf{X} \]

Equation 2-30

**Figure 8. Conceptual model of modal analysis**

In this equation it is apparent that the mode-shape matrix \( \mathbf{\Phi} \) serves to transform from the generalized coordinates \( \mathbf{X} \) to the geometric coordinates \( \mathbf{x} \). These mode-amplitude generalized coordinates are called the normal coordinates of the structure. Because the mode-shape matrix \( \mathbf{\Phi} \) for a system with \( N \) degrees of freedom consists of \( N \) independent modal vectors, it is nonsingular and can be inverted. Thus, it is always possible to solve equation 2-30 directly for the normal-coordinate amplitudes \( \mathbf{X} \) associated with any given displacement vector \( \mathbf{x} \). However, the orthogonality property makes it unnecessary to solve any simultaneous equations in evaluating \( \mathbf{X} \). To evaluate any arbitrary normal coordinate \( X_n \), equation 2-30 can be multiplied by the product of the transpose of the corresponding modal vector and the mass matrix \( \varphi_n^T \mathbf{m} \) thus

\[ \mathbf{\Phi}^T \mathbf{m} \mathbf{x} = \varphi_n^T \mathbf{m} \mathbf{\Phi} \]

Equation 2-31

The right-hand side of this equation can be expanded to give

\[ \mathbf{\Phi}^T \mathbf{m} \mathbf{\Phi} \mathbf{X} = \varphi_1^T \mathbf{m} \varphi_1 \mathbf{X}_1 + \varphi_2^T \mathbf{m} \varphi_2 \mathbf{X}_2 + \ldots + \varphi_N^T \mathbf{m} \varphi_N \mathbf{X}_N \]

Equation 2-32

However, all terms of this series vanish except that corresponding to \( \varphi_n \) because of the orthogonality property with respect to mass; thus introducing this one term on the right side of equation 2-32 gives

\[ \varphi_n^T \mathbf{m} \mathbf{x} = \varphi_n^T \mathbf{m} \varphi_n \mathbf{X}_n \]

Equation 2-33

from which

\[ X_n = \frac{\varphi_n^T \mathbf{m} \mathbf{x}}{\varphi_n^T \mathbf{m} \varphi_n} \]

Equation 2-34

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Of course, each of the normal coordinates $X_1...X_N$ is given by an expression of this type. This process is very similar to the way one derives Fourier coefficients.

**Uncoupled Equations Of Motion: Undamped**

The orthogonality properties of the normal coordinates now may be used to simplify the equations of motion of the MDOF system. In general form these equations are given by equation 2-22; for the undamped system they become

$$m\ddot{X} + kX = p(t)$$  \hspace{1cm} \text{Equation 2- 35}

Introducing equation 2-30 and its second time derivative $\ddot{X} = \Phi\dddot{X}$ (noting that the mode shapes do not change with time) leads to

$$m\Phi\dddot{X} + k\Phi X = p(t)$$  \hspace{1cm} \text{Equation 2- 36}

If equation 2-36 is pre-multiplied by the transpose of the nth mode-shape vector $\varphi_n^T$, it becomes

$$\varphi_n^Tm\Phi\dddot{X} + \varphi_n^Tk\Phi X = \varphi_n^Tp(t)$$  \hspace{1cm} \text{Equation 2- 37}

but if the two terms on the left-hand side are expanded as shown in equation 2-32, all terms except the nth will vanish because of the mode-shape orthogonality properties; hence the result is

$$\varphi_n^Tm\varphi_n\dddot{X}_n + \varphi_n^Tk\varphi_nX_n = \varphi_n^Tp(t)$$  \hspace{1cm} \text{Equation 2- 38}

Now new symbols will be defined as follows

$$M_n = \varphi_n^Tm\varphi_n \quad K_n = \varphi_n^Tk\varphi_n \quad P_n(t) = \varphi_n^Tp(t)$$  \hspace{1cm} \text{Equation 2- 39}

which are called the normal-coordinate generalized mass, generalized stiffness, and generalized load for mode $n$, respectively. With them equation 2-38 can be written

$$M_n\dddot{X}_n + K_nX_n = P_n(t)$$  \hspace{1cm} \text{Equation 2- 40}

which is a SDOF equation of motion for mode $n$. If equation 2-27, $k\varphi_n = \omega^2m\varphi_n$, is multiplied on both sides by $\varphi_n^T$, the generalized stiffness for mode $n$ is related to the generalized mass by the frequency of vibration

$$K_n = \omega^2M_n$$  \hspace{1cm} \text{Equation 2- 41}

(Capital letters are used to denote all normal-coordinate properties.) The procedure described above can be used to obtain an independent SDOF equation for each mode of vibration of the structure. Thus the use of the normal coordinates serves to transform the equations of motion from a set of $N$ simultaneous differential equations, which are coupled by the off-diagonal terms in the mass and stiffness matrices, to a set of $N$ independent normal-coordinate equations. The dynamic response therefore can be obtained by solving separately for the response of each normal (modal) coordinate and then superposing these by equation 2-29 to obtain the response in the original coordinates. This procedure is called the **mode-superposition** method.

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Uncoupled Equations Of Motion: Damped

Now it is of interest to examine the conditions under which this normal-coordinate transformation will also serve to uncouple the damped equations of motion. These equations (equation 2-22) are

\[ m\ddot{x} + c\dot{x} + kx = p(t) \quad \text{Equation 2-42} \]

Introducing the normal-coordinate expression of equation 2-30 and its time derivatives and pre-multiplying by the transpose of the nth mode-shape vector \( \phi_n^T \) leads to

\[ \phi_n^T m\ddot{\Phi} + \phi_n^T c\dot{\Phi} + \phi_n^T k\Phi = \phi_n^T p(t) \quad \text{Equation 2-43} \]

It was noted above that the orthogonality conditions cause all components except the nth mode term in the mass and stiffness expressions of equation 2-37 to vanish. A similar reduction will apply to the damping expression if it is assumed that the corresponding orthogonality condition applies to the damping matrix; that is, assume that \( \phi_m^T c\phi_n = 0 \) \( m \neq n \) \[ \text{Equation 2-44} \]

In this case equation 2-40 may be written

\[ M_n\ddot{X}_n + C_n\dot{X}_n + K_nX_n = P_n(t) \quad \text{Equation 2-45} \]

or alternatively

\[ \ddot{X}_n + 2\zeta_n\omega_n\dot{X}_n + \omega_n^2X_n = \frac{P_n(t)}{M_n} \quad \text{Equation 2-46} \]

in which

\[ M_n = \phi_n^T m\phi_n, \quad C_n = \phi_n^T c\phi_n = 2\zeta_n\omega_n M_n, \quad K_n = \phi_n^T k\phi_n, \quad P_n(t) = \phi_n^T p(t) \quad \text{Equation 2-47} \]

The normal-coordinate generalized mass, stiffness, and load for the damped system are identical to those for the undamped system (equation 2-39). The generalized damping for mode \( n \), which is given by equation 2-47, is of equivalent form. The right-hand term in this equation constitutes a definition of the nth-mode damping ratio \( \zeta_n \), because the other factors in the expression are known. As noted earlier, it generally is much more convenient and physically reasonable to define the damping by the damping ratio for each mode than it is to try to evaluate the coefficients of the damping matrix \( c \).

Conditions for Damping Orthogonality

In this derivation of the normal-coordinate equations of motion, it has been assumed that the normal-coordinate transformation serves to uncouple the damping forces in the same way that it uncouples the inertia and elastic forces. The vibration mode shapes in the damped system will then be the same as the undamped mode shapes. It is now useful to consider the conditions under which this uncoupling will occur, that is, the form of damping matrix to which equation 2-44 applies. Rayleigh showed that a damping matrix of the form

\[ c = a_0 m + a_1 k \quad \text{Equation 2-48} \]

in which \( a_0 \) and \( a_1 \) are arbitrary proportional factors, will satisfy the orthogonality condition equation 2-44. This is readily demonstrated by applying orthogonality operation on both sides.
of equation 2-48; thus it is evident that a damping matrix proportional to the mass and/or stiffness matrices will permit uncoupling the equations of motion. However it was demonstrated earlier that an infinite number of matrices formed from the mass and stiffness matrices also satisfy the orthogonality condition. Therefore the damping matrix can also be made up of the combinations of these. In general, the orthogonal damping matrix may be of the form

$$c = m \sum_b a_b [m^{-1} k]^b = \sum_b c_b$$

Equation 2-49

in which as many terms may be included as desired. Rayleigh damping (equation 2-48) obviously is contained in equation 2-49; however, by including additional terms in this equation it is possible to obtain a greater degree of control over the modal damping ratios resulting from the damping matrix. With this type of damping matrix it is possible to compute the damping influence coefficients necessary to provide a decoupled system having any desired damping ratios in any specified number of modes. For each mode \(n\), the generalized damping is given by equation 2-47:

$$\omega_n \xi_n \phi_n^TM \phi_n = \omega_n^2 \phi_n^T m[k^{-1}] \phi_n$$

Equation 2-50

But if \(c\) is given by equation 2-49, the contribution of term \(b\) of the series to the generalized damping is

$$C_{nb} = \phi_n^T c_b \phi_n = a_b [\phi_n^T m[k^{-1}] \phi_n]$$

Equation 2-51

Now if equation 2-27 \((k \phi_n = \omega_n^2 m \phi_n)\) is pre-multiplied on both sides by \(\phi_n^T k^{-1}\), the result is

$$\phi_n^T k^{-1} k \phi_n = \omega_n^2 \phi_n^T m[k^{-1}] \phi_n = \omega_n^2 \phi_n^T M_n$$

Equation 2-52

By operations equivalent to this, it can be shown that

$$\phi_n^T m[k^{-1}] \phi_n = \omega_n^2 \phi_n^T M_n$$

Equation 2-53

and consequently

$$C_{nb} = \omega_n^2 a_b \phi_n^T M_n$$

Equation 2-54

On this basis, the damping matrix associated with any mode \(n\) is

$$C_n = \sum_b C_{nb} = \sum_b a_b \omega_n^{2b} M_n = 2 \xi_n \omega_n M_n$$

Equation 2-55

from which

$$\xi_n = \frac{1}{2 \omega_n} \sum_b a_b \omega_n^{2b}$$

Equation 2-56

Equation 2-56 provides the means for evaluating the constants \(a_b\) to give the desired damping ratios in any specified number of modes. As many terms must be included in the series as there are specified modal damping ratios; then the constants can be determined from the resulting set of simultaneous equations. In principle, the values of \(b\) can lie anywhere in the range \(-\infty < b < \infty\), but in practice it is desirable to select values as near to zero as possible. For example, to evaluate the coefficients to provide for three specified damping ratios, the equations resulting from equation 2-55 would be

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In general, the corresponding relationship may be written symbolically as

$$\zeta = \frac{1}{2} Qa$$  

Equation 2-58

where $Q$ is a square matrix involving different powers of the modal frequencies. Equation 2-58 can then be solved for the coefficients $a$

$$a = 2Q^{-1} \zeta$$  

Equation 2-59

and finally the damping matrix can be obtained from equation 2-49.

It is of interest to note in equation 2-56 (or 2-57) that when the damping matrix is proportional to the mass matrix ($c = a_0m$; that is, $b = 0$), the damping ratio is inversely proportional to the frequency of vibration; thus the higher modes of a structure will have very little damping. Similarly, where the damping is proportional to the stiffness matrix ($c = a_1k$; that is, $b = 1$), the damping ratio is directly proportional to the frequency; and the higher modes of the structure will be very heavily damped.

A second method is available for evaluating the damping matrix associated with any given set of modal damping ratios. In principle, the procedure can be explained by considering the complete diagonal matrix of generalized damping coefficients, which may be obtained by pre- and post-multiplying the damping matrix by the mode-shape matrix:

$$C = \Phi^T c \Phi = 2 \begin{bmatrix} \zeta_1 \omega_1 M_1 & 0 & 0 & \cdots \\ 0 & \zeta_2 \omega_2 M_2 & 0 & \cdots \\ 0 & 0 & \zeta_3 \omega_3 M_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$  

Equation 2-60

It is evident from this equation that the damping matrix can be obtained by pre- and post-multiplying $C$ by the inverse of the mode-shape matrix or its transpose:

$$[\Phi^T]^{-1} C \Phi^{-1} = [\Phi^T]^{-1} \Phi^T c \Phi \Phi^{-1} = c$$  

Equation 2-61

Thus for any specified set of modal damping ratios $\zeta_n$, the generalized damping coefficients $C$ can be evaluated, as indicated in equation 2-60, and then the damping matrix $c$ evaluated as in equation 2-61.

In practice, however, this is not a very convenient procedure because the inversion of the mode-shape matrix is a large computational job. Instead, it is useful to take advantage of the orthogonality properties of the mode shapes relative to the mass matrix. The diagonal generalized-mass matrix of the system is obtained by pre- and post-multiplying the mass matrix by the complete mode-shape matrix:

$$M = \Phi^T m \Phi$$  

Equation 2-62
Pre-multiplying this by the inverse of the generalized-mass matrix then gives

\[ I = M^{-1} = \left[ M^{-1} \Phi^T m \right] \Phi = \Phi^{-1} \Phi \]  

Equation 2-63

from which it is evident that the mode-shape-matrix inverse is

\[ \Phi^{-1} = M^{-1} \Phi^T m \]  

Equation 2-64

The damping matrix now is given by substituting equation 2-64 into equation 2-61:

\[ c = [m \Phi M^{-1} ] C [M^{-1} \Phi^T m] \]  

Equation 2-65

Since \( c_n = 2 \xi_n \omega_n M_n \), the elements of the diagonal matrix obtained as the product of the three central diagonal matrices in equation 2-65 are given by

\[ \xi_n = \frac{2 \xi_n \omega_n}{M_n} \]  

Equation 2-66

and equation 2-65 may be written

\[ c = m \Phi \xi \Phi^T m \]  

Equation 2-67

where is the diagonal matrix of elements \( \xi_n \). In practice it is more convenient to note that each modal damping ratio provides an independent contribution to the damping matrix, as follows:

\[ c_n = m \Phi \xi_n \Phi^T m \]  

Equation 2-68

Thus the total damping matrix is obtained as the sum of the modal contributions

\[ c = \sum_{n=1}^{N} c_n = m \left[ \sum_{n=1}^{N} \Phi_n \xi_n \Phi_n^T \right] m \]  

Equation 2-69

By substituting from equation 2-66 this may be written

\[ c = m \left[ \sum_{n=1}^{N} \frac{2 \xi_n \omega_n}{M_n} \Phi_n \Phi_n^T \right] m \]  

Equation 2-70

In this equation, the contribution to the damping matrix from each mode is proportional to the modal damping ratio; thus any undamped mode will contribute nothing to the damping matrix. In other words, only those modes specifically included in the formation of the damping matrix will have any damping; all other modes will be undamped.

At this point, it is well to consider under what circumstances it may be desirable to evaluate the elements of a damping matrix explicitly, as by equation 2-49 or 2-70. It has been noted that the modal damping ratios are the most effective measures of the damping in the system when the analysis is to be carried out by the mode-superposition method. Hence the damping matrix will be needed in explicit form primarily when the dynamic response is to be obtained by some other analysis procedure, e.g., step-by-step integration of a nonlinear system.

**Damping Coupling**

In the foregoing paragraphs, it has been emphasized that where the damping matrix of the structure is of a form which satisfies the modal orthogonality conditions, the transformation to the undamped modal coordinates leads to a set of uncoupled equations.
Since the response of the system can then be obtained by superposing the responses given by these SDOF equations, this decoupling is a major advantage of the normal coordinates. It was mentioned earlier, however, that these coordinates have another major advantage which can be equally important: the essential dynamic response often is associated with the lowest few modal coordinates, which means that a good approximation to the response can be often obtained with a drastically reduced number of coordinates.

Where the dynamic response is contained in only a few of the lower modes, it clearly will be advantageous to apply the normal-coordinate transformation, even with structures for which the damping matrix does not satisfy the orthogonality condition. In this case, the generalized damping matrix will not be diagonal; that is, the modal equations will be coupled by the generalized damping forces. Consequently, the response must be obtained by integrating these equations simultaneously rather than individually. However, this integration can be carried out by step-by-step methods, and certainly it is more efficient to perform the integration for a few coupled normal-coordinate equations than for the original coupled-equation system.

An alternative procedure would be to solve the complex eigenproblem (which results when the damping matrix is of general form) and then to obtain an uncoupled set of equations by transforming to the damped modal coordinates. However, the evaluation of the damped mode shapes requires much more computation than the undamped eigenproblem solution does; the problem is of order $2N$ for a system with $N$ degrees of freedom because a phase angle must be evaluated for each degree of freedom as well as its relative amplitude. For this reason the use of the undamped mode shapes generally is more efficient. The complex eigenproblem is discussed in more detail under “Methods of complex response”.

**SUMMARY OF THE MODE-SUPERPOSITION PROCEDURE**

The normal-coordinate transformation, which serves to change the set of $N$ coupled equations of motion of a MDOF system into a set of $N$ uncoupled equations, is the basis of the mode-superposition method of dynamic analysis. This method can be used to evaluate the dynamic response of any linear structure for which the displacements have been expressed in terms of a set of $N$ discrete coordinates and where the damping can be expressed by modal damping ratios. The procedure consists of the following steps.

1. **Equations Of Motion**
   For this class of system, the equations of motion may be expressed [Eqs. (10-13)] as
   \[
   m\ddot{x} + c\dot{x} + kx = p(t)
   \]

2. **Mode Shape And Frequency Analysis**
   For undamped, free vibrations, this matrix equation can be reduced to the eigenvalue equation [Eq. (12-4)]:
   \[
   (k - \omega^2m)x = 0
   \]
   from which the vibration mode-shape matrix $\Phi$ and frequency vector $\omega$ can be determined.

3. **Generalized Mass And Load**
   With each mode-shape vector $\varphi_n$ being used in turn, the generalized mass and generalized load for each mode can be computed.
   \[
   M_n = \varphi_n^Tm\varphi_n, \quad P_n = \varphi_n^Tp(t)
   \]

4. **Uncoupled Equations Of Motion**
   The equation of motion for each mode can then be written, using the generalized mass and force for the mode together with the modal frequency $\omega_n$ and a specified value of the modal damping ratio $\zeta_n$ as follows
5. Modal Response To Loading  The result of step 4 is a set of \( N \) independent equations of motion, one for each mode of vibration. These SDOF equations can be solved by any suitable method, depending on the type of loading. The general response expression given by the Duhamel integral [Eq. (7-14)] for each mode is

\[
Y_n(t) = \frac{1}{M_n \omega_{Dn}} \int_0^t P_n(\tau) e^{-\xi_n \omega_{n} t} \sin(\omega_{Dn} t - \tau) d\tau
\]

6. Modal Free Vibrations. Equation (7-14) is applicable for a system which is at rest at time \( t = 0 \). If the initial velocity and displacement are not zero, a free vibration response must be added to the Duhamel integral expression for each mode. The general damped free-vibration response is given [Eq. (3-26)] for each mode by

\[
Y_n(t) = e^{-\xi_n \omega_{n} t} \left[ \hat{Y}_n(0) + \hat{Y}_n(0) \xi_n \omega_n \omega_{Dn} \sin(\omega_{Dn} t + \hat{Y}_n(0) \cos \omega_{Dn} t) \right]
\]

where \( \hat{Y}_n(0) \) and \( \hat{Y}_n(0) \) represent the initial modal displacement and velocity. These can be obtained from the specified initial displacement \( x(0) \) and velocity \( \dot{x}(0) \) expressed in the original geometric coordinates as follows for each modal component [Eq. (13-5)]:

\[
Y_n(0) = \frac{\Phi^T_n m x(0)}{M_n} \quad \hat{Y}_n(0) = \frac{\Phi^T_n m \dot{x}(0)}{M_n}
\]

7. Displacement Response In Geometric Coordinates. When the response for each mode \( Y_n(t) \) has been determined from Eq.(2-14) and/or Eq. (2-26), the displacements expressed in geometric coordinates are given by the normal-coordinate transformation, Eq. (13-2):

\[
x(t) = \Phi Y(t)
\]

Equation (13-2) may also be written

\[
x(t) = \Phi_1 Y_1(t) + \Phi_2 Y_2(t) + \Phi_3 Y_3(t) + \cdots
\]

that is, it merely represents the superposition of the various modal contributions; hence the name mode-superposition method. It should be noted that for most types of loadings the contributions of the various modes generally are greatest for the lowest frequencies and tend to decrease for the higher frequencies. Consequently, it usually is not necessary to include all the higher modes of vibration in the superposition process [Eq. (13-2)]; the series can be truncated when the response has been obtained to any desired degree of accuracy. Moreover, it should be kept in mind that the mathematical idealization of any complex structural system also tends to be less reliable in predicting the higher modes of vibration; for this reason, too, it is well to limit the number of modes considered in a dynamic-response analysis.

8. Elastic Force Response. The displacement history of the structure may be considered to be the basic measure of its response to dynamic loading. In general, other response parameters such as stresses or forces developed in various structural components can be evaluated directly from the displacements. For example, the elastic forces \( f_i \), which resist the deformation of the structure are given directly [Eq. (10-6)] by

\[
f_i(t) = kx(t) = k\Phi Y(t)
\]
An alternative expression for the elastic forces may be useful in cases where the frequencies and mode shapes have been determined from the flexibility form of the eigenvalue equation [Eq. (12-17)]. Writing Eq. (13-43) in terms of the modal contributions

\[ f_j(t) = k_1 \phi_1 Y_1(t) + k_2 \phi_2 Y_2(t) + k_3 \phi_3 Y_3(t) + \cdots \]

and substituting Eq. (12-39) leads to

\[ f_j(t) = \omega_1^2 m_1 \phi_1 Y_1(t) + \omega_2^2 m_2 \phi_2 Y_2(t) + \omega_3^2 m_3 \phi_3 Y_3(t) + \cdots \]

Writing the series in matrix form gives

\[ \begin{bmatrix} f_j(t) \\ \vdots \end{bmatrix} = m \Phi \begin{bmatrix} \omega_1^2 Y_1(t) \\ \vdots \end{bmatrix} \]

where \( \begin{bmatrix} \omega_n^2 Y_n(t) \end{bmatrix} \) represents a vector of modal amplitudes each multiplied by the square of its modal frequency. In Eq. (13-44) the elastic force associated with each modal component has been replaced by an equivalent modal inertia-force expression. The equivalence of these expressions was demonstrated from the equations of free-vibration equilibrium [Eq. (13-29)]; however, it should be noted that this substitution is valid at any time, even for a static analysis. Because each modal contribution is multiplied by the square of the modal frequency in Eq. (13-44), it is evident that the higher modes are of greater significance in defining the forces in the structure than they are in the displacements. Consequently, it will be necessary to include more modal components to define the forces to any desired degree of accuracy than to define the displacements.

**Method of complex response**

It has become common practice with newer, more efficient computer codes to analyze a damped system via method of complex response. As mentioned earlier, the computational effort is greater since one must tract a complex value (two values) instead of a real value (one value). However, much of the economies of calculation discussed earlier still apply. This method is employed in a variety of well-known codes such as LUSH, FLUSH, SASSI, and others. The main idea is to exploit the use of superposition and harmonic representation of earthquake shaking.