Finite Element Transient Analysis

version 2.1.00

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User's Manual
7. Methodologies of Solution

Governing Equation
The governing equations of motion of a multi-degree of freedom (MDOF) system can be cast in a matrix form as,

\[ M \ddot{u}(t) + C \dot{u}(t) + Ku(t) = f(t) \]  \hspace{1cm} 7.1

where \( M, C, \) and \( K \) are the mass, damping and stiffness matrices of the dynamic system, \( u(t) \) represents the displacements of the nodal points of the system, \( f(t) \) are the nodal equivalent forces, and dots indicate differentiation with respect to time. The mass, damping and stiffness matrices are assembled from the respective element matrices as discussed in the Element Library section.

Support Conditions and External Loading
The support conditions for both 2-D and 3-D dynamic systems pertain to restrains of the degrees of freedom of individual nodes or group of nodes of the finite element mesh.

The external loading cases of the system are of the form of time histories of several types of excitations. Nodal forces and moments can be defined directly on the unrestrained nodes of the mesh. In the loading situation where known displacement and rotation time histories are prescribed, the corresponding degrees of freedom need to be restrained. To this end, program FETA automatically identifies the nodes subjected to forced motion and restrains the appropriate degrees of freedom. Ground motion in the form of known time histories of displacement, velocity and acceleration, is also accommodated.

Frequency Domain Solution
For frequency domain analysis the solution is obtained in the form of a complex frequency response by letting the forcing function to be of the form

\[ f(t) = f_0 e^{i\omega t} \]  \hspace{1cm} 7.2

The steady state response to this excitation has the same frequency as the exciting force and is expressed in terms of the complex frequency response, \( U \), as,

\[ u = U e^{i\omega t} \]  \hspace{1cm} 7.3

Substitution of equation 7.3 and its derivatives into equation 7.1 gives the governing equation adopted in FETA for frequency domain analysis, as
The damping matrix \( C \) is evaluated on the basis of the hysteretic damping of the material. Equation 7.4 is solved for a series of frequencies \( \omega \) for the complex frequency response \( U \). Subsequently, the time history of the displacements can be obtained through a Fourier transformation of the complex response to the time domain.

In the event that soil-structure models are considered, the far field is assumed to be a horizontally layered region. Each layer is a linearly elastic, isotropic medium. The far field extends to infinity in all horizontal directions, and its surface is free of tractions. At the base of the far field, two possibilities are available in FETA: rigid bedrock or a homogeneous, linearly elastic, isotropic half-space. The representation of the far field as a hyperelement is accomplished in the frequency domain by the finite element method. An expression of far-field motion in terms of semi-discrete wave modes (vertically discrete, horizontally continuous) is the basis of the formulation. Brief descriptions of the hyperelements implemented in FETA for two-dimensional as well as three-dimensional frequency domain computations are given in chapter Element Library.

**Time Domain Solution**

For direct time domain analysis the solution the governing equation of motion 7.1 is obtained in the time marching scheme derived by the Newmark-\( \beta \) method. The mass, damping and stiffness matrices are assembled from the corresponding element matrices. The external excitation is in the form of time histories of nodal forces, or displacements. In the event that infinite soil regions are considered in the soil-structure model, the boundary of the finite domain is complemented by absorbing elements consisting of uniformly distributed dashpots, as defined in the Element Library section.

**Newmark-\( \beta \) Method**

The direct numerical integration in time of second order differential equations of motion may be formulated as some type of finite difference scheme. To this end, program FETA implements the Newmark-\( \beta \) method for the computation of direct time domain solutions of the considered elastodynamic systems. This method is preferred because of its controlled stability characteristics. Furthermore, this method accommodates iterative schemes of the solution of nonlinear dynamic systems. The first step of any direct numerical integration method is the definition of a time knot sequence, \( t_j \). On the basis of this time knot sequence, Newmark presented equations for approximating the displacement and velocity at time \( t_{j+1} \) which for a multi degree of freedom system are as follows:

\[
\Delta t_j = t_{j+1} - t_j \\
\begin{align*}
\mathbf{u}_{j+1} &= \mathbf{u}_j + \Delta t \ddot{\mathbf{u}}_j + \frac{\Delta t^2}{2} \left[ (1 - \beta_2) \dddot{\mathbf{u}}_j + \beta_2 \dddot{\mathbf{u}}_{j+1} \right] \\
\dddot{\mathbf{u}}_{j+1} &= \dddot{\mathbf{u}}_j + \left[ (1 - \beta_1) \dddot{\mathbf{u}}_j + \beta_1 \dddot{\mathbf{u}}_{j+1} \right] \Delta t_j
\end{align*}
\]
where $\beta_1$ and $\beta_2$ are the Newmark-$\beta$ coefficients, and subscripts indicated the time step. Equation #.2 together with the dynamic Equation #.1 are used to determine the three unknowns $\ddot{u}_{j+1}$, $\dot{u}_{j+1}$, $u_{j+1}$. First the displacement at step $j+1$ is evaluated as,

$$u_{j+1} = \left( M + C\beta_1\Delta t + \frac{K\beta_2\Delta t^2}{2} \right)^{-1} \left\{ \frac{-\beta_2\Delta t^2}{2} f_{j+1} - C\left( \ddot{u}_j + \Delta t(1 - \beta_1)\dddot{u}_j \right) \right\} \left( \dddot{u}_j - \beta_1\Delta t \left( u_j + \Delta t\dddot{u}_j + \frac{\Delta t^2}{2}(1 - \beta_2)\dddot{u}_j \right) \right\} + M \left( u_j + \Delta t\dddot{u}_j + \frac{\Delta t^2}{2}(1 - \beta_2)\dddot{u}_j \right)$$

Subsequently, the velocity and acceleration are computed by solving the system of Equations 7.5.

The parameter $\beta_1$ introduces numerical (algorithmic) damping within the step $\Delta t_j$. If $\beta_1$ is taken to be equal to 0.5 then no numerical damping is considered. Values of $\beta_1$ less than 0.5 produce negative damping. If $\beta_1$ is taken greater than 0.5 then positive artificial damping is introduced. The parameter $\beta_2$ controls the variation of acceleration within the time step. For example if $\beta_2$ is taken equal to 0 then the acceleration is constant within the step. The method is unconditionally stable for values of the $\beta$ coefficients satisfying the inequality $\beta_2 \geq \beta_1 \geq 0.5$.

Solution of the System of Equations.
The dynamic governing equations are solved in the time marching scheme introduced by the Newmark-$\beta$ method. First a sequence of equally spaced time knots is defined on the time axis. Subsequently, the equations of motion can be written for step $j$ and solved for step $j+1$ in terms of the field variables at step $j$ and $j+1$ as,

$$Du_{j+1} = \vec{f}_{j+1} + b_j$$

where
The system of equations 7.7 is solved for the unknown variables at step $j+1$. To this end, a real arithmetic frontal solver is developed on the basis of the work presented by Irons [1]. This algorithm is based on a Gauss elimination scheme. However, it is more involved than the standard band-matrix algorithms, but is more efficient especially in the case where higher order elements are considered. In addition, for the present time marching scheme, an efficient technique to accommodate multiple right hand sides is implemented. For linear applications the coefficient matrix, $D$, needs to be evaluated once and the system of equations 7.7 is solved for the different right hand sides defined by equations 7.9 and 7.10. For nonlinear elastic applications, the coefficient matrix $D$ is time dependent and is evaluated at every time step, provided that the stress-strain relationships are defined.
8. ELEMENT LIBRARY

This section describes the finite elements and their formulation as implemented in FETA. Tables 8.1-8.5 present a list of all available elements and their modeling features.

Table 8.1 Structural Elements

<table>
<thead>
<tr>
<th>Description</th>
<th>Nodes</th>
<th>Global Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D Axial force element for truss modeling</td>
<td>2</td>
<td>x  y z θx θy θz</td>
</tr>
<tr>
<td>3D Axial force element for truss modeling</td>
<td>2</td>
<td>* * *</td>
</tr>
<tr>
<td>2D Beam element for framed structures</td>
<td>2</td>
<td>* *</td>
</tr>
<tr>
<td>3D Beam element for framed structures</td>
<td>2</td>
<td>* * * * * *</td>
</tr>
<tr>
<td>2D Linear Spring-Damper System</td>
<td>2</td>
<td>*</td>
</tr>
<tr>
<td>3D Linear Spring-Damper System</td>
<td>2</td>
<td>* *</td>
</tr>
<tr>
<td>2D Concentrated Mass</td>
<td>1</td>
<td>*</td>
</tr>
<tr>
<td>3D Concentrated Mass</td>
<td>1</td>
<td>* * * * * * *</td>
</tr>
<tr>
<td>2D Concentrated Stiffness</td>
<td>1</td>
<td>*</td>
</tr>
<tr>
<td>3D Concentrated Stiffness</td>
<td>1</td>
<td>* * * * * * *</td>
</tr>
<tr>
<td>3D Horizontal Plate</td>
<td>8</td>
<td>* *</td>
</tr>
<tr>
<td>3D General Shell Element</td>
<td>8</td>
<td>* * * * * * *</td>
</tr>
</tbody>
</table>

Table 8.2 Soil/Solid Elements

<table>
<thead>
<tr>
<th>Description</th>
<th>Nodes</th>
<th>Global Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D Constant Strain Triangle</td>
<td>3</td>
<td>x  y z θx θy θz</td>
</tr>
<tr>
<td>2D Linear Strain Triangle</td>
<td>6</td>
<td>*</td>
</tr>
<tr>
<td>2D Constant Strain Quadrilateral</td>
<td>4</td>
<td>*</td>
</tr>
<tr>
<td>2D Linear Strain Quadrilateral</td>
<td>8</td>
<td>*</td>
</tr>
<tr>
<td>3D Linear Strain Tetrahedron</td>
<td>10</td>
<td>* *</td>
</tr>
<tr>
<td>3D Linear Strain Pentahedron</td>
<td>15</td>
<td>* *</td>
</tr>
<tr>
<td>3D Linear Strain Hexahedron</td>
<td>20</td>
<td>* *</td>
</tr>
</tbody>
</table>
### Table 8.3 Special Soil Elements

<table>
<thead>
<tr>
<th>Description</th>
<th>Nodes</th>
<th>Global Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>2D Vertical Transmitting Boundaries</td>
<td>varied</td>
<td>*</td>
</tr>
<tr>
<td>3D Vertical Cylindrical Transmitting</td>
<td>varied</td>
<td>*</td>
</tr>
<tr>
<td>Bound.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2D Absorbing Boundary</td>
<td>2 or 3</td>
<td>*</td>
</tr>
<tr>
<td>3D Absorbing Boundary</td>
<td>6 or 8</td>
<td>*</td>
</tr>
</tbody>
</table>

### Table 8.4 Constraining Elements

<table>
<thead>
<tr>
<th>Description</th>
<th>Nodes</th>
<th>Global Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>2D Rigid Link</td>
<td>3</td>
<td>*</td>
</tr>
<tr>
<td>3D Rigid Link</td>
<td>3</td>
<td>*</td>
</tr>
<tr>
<td>2D Rigid Footing</td>
<td>3</td>
<td>*</td>
</tr>
<tr>
<td>3D Rigid Footing</td>
<td>3</td>
<td>*</td>
</tr>
<tr>
<td>2D Frictional Interface</td>
<td>3</td>
<td>*</td>
</tr>
</tbody>
</table>

### Table 8.5 Fluid Elements

<table>
<thead>
<tr>
<th>Description</th>
<th>Nodes</th>
<th>Global Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Q</td>
</tr>
<tr>
<td>2D Quadrilateral pressure element</td>
<td>4</td>
<td>*</td>
</tr>
<tr>
<td>2D Triangular pressure element</td>
<td>3</td>
<td>*</td>
</tr>
<tr>
<td>2D Fluid-Solid Interface element</td>
<td>3</td>
<td>*</td>
</tr>
<tr>
<td>2D Free surface fluid element</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
2-D Uniaxial Force Element

The two dimensional uniaxial force element is a tension-compression element with two nodes. Each node has two translational degrees of freedom, in the global X and Y directions. The element can take only nodal forces and bending is not allowed. This element is used to model 2-D truss members.

Shape Functions

The shape functions for the 2-D uniaxial element are

\[ N_1(\xi) = \frac{1}{2} (1 - \xi) \quad N_2(\xi) = \frac{1}{2} (1 + \xi) \]

Stiffness Matrix

The stiffness matrix of the 2D uniaxial tension-compression member is given in an explicit form as:

\[
[K] = \frac{E A}{L} \begin{bmatrix}
1^2 & \text{sym.} \\
\text{lm} & m^2 \\
-l^2 & -\text{lm} & 1^2 \\
-\text{lm} & -m^2 & \text{lm} & m^2
\end{bmatrix}
\]

where \( l \) and \( m \) are the directional cosines of the element defined as:
Mass Matrix
The mass matrix is evaluated according to the consistent formulation and is given in an explicit form as,

\[
[M] = \frac{\rho A L}{6} \begin{bmatrix}
2 & \text{sym.} \\
0 & 2 \\
1 & 0 & 2 \\
0 & 1 & 0 & 2
\end{bmatrix}
\]

where \( \rho \) is the mass density

Damping Matrix
For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

where a and b are user defined constants.
3-D Uniaxial Force Element

The three dimensional uniaxial force element is a tension-compression element with two nodes. Each node has three translational degrees of freedom, in the global X and Y and Z directions. The element can take only nodal forces and bending is not allowed. This element is used to model 3-D truss members.

Shape Functions
The shape functions for the 3-D uniaxial element are

\[ N_1(\xi) = \frac{1}{2}(1 - \xi) \quad N_2(\xi) = \frac{1}{2}(1 + \xi) \]

Stiffness Matrix
The stiffness matrix of the 3-D uniaxial tension-compression member is given in an explicit form as:

\[
[K] = \frac{EA}{L} \begin{bmatrix}
 l^2 & l m & m^2 & \text{symm} \\
 l m & m n & n^2 \\
 -l^2 & -l m & -m n & -l^2 \\
 -l m & -m^2 & -m n & l m & m^2 \\
 -l n & -m n & -n^2 & l n & m n & n^2
\end{bmatrix}
\]
where \( l, m, \) and \( n \) are the directional cosines of the element defined as:

\[
l = \frac{X_2 - X_1}{L}, \quad m = \frac{Y_2 - Y_1}{L}, \quad n = \frac{Z_2 - Z_1}{L}
\]

\[
L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}
\]

**Mass Matrix**

The mass matrix is evaluated according to the consistent formulation and is given in an explicit form as,

\[
[M] = \frac{\rho AL}{6}
\]

where \( \rho \) is the mass density

\[
[M] = \begin{bmatrix}
2 & 0 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 \\
0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 2
\end{bmatrix}
\]

\[
\text{where } \rho \text{ is the mass density}
\]

**Damping Matrix**

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

where \( a \) and \( b \) are user defined constants.
2-D Elastic Beam Element

The two dimensional elastic beam element is a straight line element with two nodes. Each node has two translational, and one rotational degrees of freedom in the global coordinate system. The element takes either nodal forces or displacements. This element is used to model plane frame members.

Shape Functions

The shape functions for the 2-D beam element are

\[ N_1(\xi) = \frac{1}{2}(1 - \xi) \quad N_2(\xi) = 2\xi^3 - 3\xi^2 + 1 \quad N_3(\xi) = L(\xi^3 - 2\xi^2 + \xi) \]

\[ N_4(\xi) = \frac{1}{2}(1 + \xi) \quad N_5(\xi) = -2\xi^3 + 3\xi^2 \quad N_6(\xi) = L(\xi^3 - \xi^2) \]

and the displacement functions \( u_x \) and \( u_y \) are interpolated in terms of the nodal displacements \( u_i \) as

\[ u_x(\xi, t) = N_1(\xi)u_1(t) + N_4(\xi)u_4(t) \]

\[ u_y(\xi, t) = N_2(\xi)u_2(t) + N_3(\xi)u_3(t) + N_5(\xi)u_5(t) + N_6(\xi)u_6(t) \]
Stiffness Matrix

The stiffness matrix of the 2D elastic beam member is given in an explicit form with respect to the element coordinate system as:

\[
[K] = \frac{E}{L} \begin{bmatrix}
A & 12I & \text{symm} \\
0 & \frac{12I}{L} & 0 \\
0 & \frac{6I}{L} & 0 \\
0 & \frac{2I}{L} & 0 \\
0 & \frac{12I}{L} & 0 \\
0 & \frac{6I}{L} & 0 \\
0 & \frac{2I}{L} & 0 \\
\end{bmatrix}
\]

where \( A \) is the cross sectional area, and \( I \) is the moment of inertia.

Mass Matrix

The mass matrix is evaluated according to the consistent formulation and is given in an explicit form with respect to the element coordinate system as,

\[
[M] = \frac{\rho AL}{420} \begin{bmatrix}
140 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 156 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 22L & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4L^2 & 0 & 0 & 0 & 0 \\
70 & 0 & 0 & 0 & 140 & 0 & 0 & 0 \\
0 & 54 & 13L & 0 & 0 & 156 & 0 & 0 \\
0 & -13L & -3L^2 & 0 & -22L & 0 & 4L^2 & 0 \\
\end{bmatrix}
\]

where \( \rho \) is the mass density.

Damping Matrix

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

where \( a \) and \( b \) are user defined constants.

3-D Elastic Beam Element
The three dimensional elastic beam element is a straight line element with two nodes. Each node has three translational and three rotational degrees of freedom, in the global coordinate system. The element takes either nodal forces or displacements. This element is used to model space frame members.

Shape Functions
The shape functions for the 3-D beam element are

\[ N_1(\xi) = 0.5(1 - \xi), \quad N_2(\xi) = 0.5(1 + \xi) \]
\[ N_3(\xi) = \frac{1}{2} \left(1 - \frac{\xi}{2}(3 - \xi^2)\right), \quad N_4(\xi) = \frac{1}{2} \left(1 + \frac{\xi}{2}(3 - \xi^2)\right) \]
\[ N_5(\xi) = 0.5(1 - \xi)(1 - \xi^2), \quad N_6(\xi) = 0.5(1 + \xi)(1 - \xi^2) \]

and the displacement functions \( u_x, u_y, u_z, \) and \( \theta_x \) are interpolated in terms of the nodal displacements \( u_i \) as

\[ u_x(\xi, t) = u_1(t)N_1(\xi) + u_7(t)N_2(\xi) \]
\[ u_y(\xi, t) = u_2(t)N_3(\xi) + u_8(t)N_4(\xi) + \frac{L}{8}(u_6(t)N_5(\xi) - u_{12}(t)N_6(\xi)) \]
\[ u_z(\xi, t) = u_3(t)N_3(\xi) + u_9(t)N_4(\xi) - \frac{L}{8}(u_5(t)N_5(\xi) - u_{11}(t)N_6(\xi)) \]
\[ \theta_x(\xi, t) = u_4(t)N_1(\xi) + u_{10}(t)N_2(\xi) \]
Stiffness Matrix
The stiffness matrix of the 3-D elastic beam member is given in an explicit form with respect to the element coordinate system as:

\[
[K] = \frac{E}{L^3} \begin{bmatrix}
  r_1 I_z & 0 & 12I_z \\
  0 & 0 & 12I_y \\
  0 & 0 & 0 & r_2 L^2 I_y \\
  0 & 0 & -6L I_y & 0 & 4L^2 I_y \\
  0 & 6L I_z & 0 & 0 & 0 \\
  0 & -12I_z & 0 & 0 & 0 & r_1 I_z \\
  0 & 0 & -12I_y & 0 & 6L I_y & 0 & 0 & 0 & 12I_y \\
  0 & 0 & 0 & -r_2 L^2 I_y & 0 & 0 & 0 & 0 & r_2 L^2 I_y \\
  0 & 0 & -6L I_y & 0 & 2L^2 I_y & 0 & 0 & 0 & 6L I_y & 0 & 4L^2 I_y \\
  0 & 6L I_z & 0 & 0 & 0 & 2L^2 I_z & 0 & -6L I_z & 0 & 0 & 0 & 4L^2 I_z
\end{bmatrix}
\]

where \( r_1 = \frac{A L^2}{I_z} \), \( r_2 = \frac{(1 + \nu) I_y}{2I_y} \), \( A \) is the cross sectional area, \( I_x \) is the torsional constant, \( I_y \), and \( I_z \) are the second moments of inertia with respect to the local axes \( y \) and \( z \), respectively, and \( \nu \) is the Poisson ratio.

Mass Matrix
The mass matrix is evaluated according to the consistent formulation and is given in an explicit form with respect to the element coordinate system as,
where $\rho$ is the mass density, and $r_g = J/A$ is the radius of gyration.

**Damping Matrix**

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

$$ [C] = a[M] + b[C] $$

where $a$ and $b$ are user defined constants.

$$ [M] = \frac{\rho A L}{420} $$

$$ \begin{bmatrix}
140 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 156 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 156 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 140r_g^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -22L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 22L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
70 & 0 & 0 & -13L & 0 & 0 & 0 & 0 & 0 \\
0 & 54 & 0 & 0 & 0 & 0 & 13L & 0 & 0 \\
0 & 0 & 54 & -13L & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 13L & -3L^2 & -3L^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -13L & 0 & 0 & 0 & -3L^2 & 0 & 0 \\
0 & 22L & 0 & 0 & 0 & 0 & -22L & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4L^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4L^2 \\
\end{bmatrix} $$

8.19
2-D Spring-Damper Element

The two dimensional spring damper element is a tension-compression element with two nodes. Each node has two translational degrees of freedom, in the global X and Y directions. The element consists of a spring and a damper of known elastic properties. It can take only nodal forces and bending is not allowed. This element is used to model 2-D elastic links.

Stiffness Matrix
The stiffness matrix of the 2D spring-damper element with respect to the global coordinate system is given in an explicit form as:

\[
[K] = k \begin{bmatrix}
l^2 & \text{sym.} \\
lm & m^2 \\
-1^2 & -lm & 1^2 \\
-lm & -m^2 & lm & m^2
\end{bmatrix}
\]

8.21

where \( k \) is the stiffness of the spring, and \( l \), and \( m \) are the directional cosines of the element defined as:

\[
l = \frac{X_2 - X_1}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}} \quad m = \frac{Y_2 - Y_1}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}
\]

8.22
Damping Matrix

For time domain analysis the damping matrix is similar to the stiffness matrix, i.e.,

\[
[C] = c \begin{bmatrix}
1^2 & \text{sym.} \\
\text{lm} & m^2 \\
-l^2 & -\text{lm} & l^2 \\
-\text{lm} & -m^2 & \text{lm} & m^2
\end{bmatrix}
\]

where \( c \) is the damping constant.
3-D Spring-Damper Element

The three dimensional spring-damper element is a tension-compression element with two nodes. Each node has three translational degrees of freedom, in the global X, Y, and Z directions. The element consists of a spring and a damper of known elastic properties. It can take only nodal forces and bending is not allowed. This element is used to model 3-D elastic links.

Stiffness Matrix

The stiffness matrix of the 3-D spring-damper element with respect to the global coordinate system is given in an explicit form as:

\[
[K] = k \begin{bmatrix}
 l^2 & m^2 & \text{sym.} \\
 lm & mn & n^2 \\
 -l^2 & -lm & -ln & 1^2 \\
 -lm & m^2 & -mn & \text{ln} & m^2 \\
 -ln & -mn & n^2 & \text{ln} & mn & n^2 \\
\end{bmatrix}
\]

where \( k \) is the spring stiffness, and \( l, m, \) and \( n \) are the directional cosines of the element defined as:

\[
l = \frac{X_2 - X_1}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}
\]
\[
m = \frac{Y_2 - Y_1}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}
\]

\[
n = \frac{Z_2 - Z_1}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}}
\]

Damping Matrix
For time domain analysis the damping matrix is similar to the stiffness matrix, i.e.,

\[
[C] = c \begin{bmatrix}
1^2 & \text{sym.} \\
\text{lm} & m^2 \\
\text{ln} & mn & n^2 \\
-l^2 & -\text{lm} & -\text{ln} & l^2 \\
-\text{lm} & m^2 & -mn & \text{lm} & m^2 \\
-\text{ln} & -mn & -n^2 & \text{ln} & mn & n^2
\end{bmatrix}
\]

where \( c \) is the damping constant.
The two dimensional concentrated mass is a point element which has two translational and one rotational degrees of freedom, in the global coordinate system. The element consists of a lumped mass which presents inertia forces and moments. This element is used to add the influence of concentrated masses to the dynamic system.

Mass Matrix
The mass matrix of the 2-D concentrated mass element with respect to the global coordinate system is a diagonal matrix given in an explicit form as:

\[
[M] = \begin{bmatrix}
  m_x & 0 & 0 \\
  0 & m_y & 0 \\
  0 & 0 & I_0
\end{bmatrix}
\]

where \(m_x\) and \(m_y\) are the masses contributing to the inertia forces in the X and Y direction, and \(I_0\) is the mass moment of inertia associated to the moment.
3-D  Concentrated Mass

The three dimensional concentrated mass is a point element which has three translational and three rotational degrees of freedom, in the global coordinate system. The element consists of a lumped mass which presents inertia forces and moments. This element is used to add the influence of concentrated masses to the 3-D dynamic system.

Mass Matrix

The mass matrix of the 3-D concentrated mass element with respect to the global coordinate system is a diagonal matrix given in an explicit form as:

\[
[M] = \begin{bmatrix}
m_x & m_y & m_z \\
& I_{ox} & \\
& & I_{oy} \\
& & & I_{oz}
\end{bmatrix}
\]

8.28

where \(m_x\), \(m_y\), and \(m_z\) are the masses contributing to the inertia forces in the X, Y, and Z direction, respectively, and \(I_{ox}\), \(I_{oy}\), \(I_{oz}\), are the mass moments of inertia in the X, Y, and Z directions, respectively.
2-D Elastic Stiffener

The two dimensional elastic stiffener is a point element which has two translational and one rotational degrees of freedom, in the global coordinate system. The element consists of a stiffener which presents elastic forces and moments. This element is used to add the influence of stiffeners to the dynamic system.

Stiffness Matrix

The stiffness matrix of the 2-D stiffener with respect to the global coordinate system is a diagonal matrix given in an explicit form as:

\[
[K] = \begin{bmatrix}
k_x & 0 \\
0 & k_y \\
k_\theta & 0
\end{bmatrix}
\]

where \(k_x\) and \(k_y\) are the stiffness coefficients contributing to the elastic forces in the X and Y direction, and \(k_\theta\) is the stiffness coefficient associated to the elastic moment.
3-D Elastic Stiffener

The three dimensional elastic stiffener is a point element which has three translational and three rotational degrees of freedom, in the global coordinate system. The element consists of stiffener which presents elastic forces and moments. This element is used to add the influence of stiffeners to the 3-D dynamic system.

Stiffness Matrix

The stiffness matrix of the 3-D elastic stiffener with respect to the global coordinate system is a diagonal matrix given in an explicit form as:

\[
[K] = \begin{bmatrix}
k_x & & \\
& k_y & \\
& & k_z \\
& k_{0x} & \\
& & k_{0y} \\
& & & k_{0z}
\end{bmatrix}
\]

where \( k_x, k_y, \) and \( k_z \) are the translational stiffness coefficients contributing to the elastic forces in the X, Y, and Z direction, respectively, and \( k_{0x}, k_{0y}, k_{0z} \) are the rotational stiffness coefficients associated to the elastic moments in the X, Y, and Z directions, respectively.
3-D Horizontal Plate - Bending Only

The element is of rectangular shape defined by eight geometric nodes and is parallel to the X-Y horizontal plane. Each node has one translational degree of freedom in the vertical Z direction, and two rotational degrees of freedom in the plane of the plate. It can take only vertical forces or bending moments.

Shape Functions
The shape functions of the 8-node plate element are

\[ N_i = \frac{1}{4} (1 + \xi_i \eta_i)(1 + \eta_i)(-1 + \xi_i + \eta_i) \quad i = 1,2,3,4 \]
\[ N_i = \frac{1}{2} (1 - \xi_i^2)(1 + \eta_i) \quad i = 5,7 \]
\[ N_i = \frac{1}{2} (1 + \xi_i)(1 - \eta_i^2) \quad i = 6,8 \]

Stiffness Matrix
The element stiffness matrix is given as
\[
[K] = \iint_{\Lambda} \left( 2[B_A]^T[E][B_A] + \frac{h^2}{6} [B_B]^T[E][B_B] \right) dA
\]  \hspace{1cm} (8.32)

where:

\([E]\) is the stress-strain matrix

\[
[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \text{sym.} \\
\nu & 1-\nu \\
\frac{1-2\nu}{2} & 1-2\nu \\
\frac{2(1.2)}{2} & \frac{1-2\nu}{2(1.2)}
\end{bmatrix}
\]  \hspace{1cm} (8.33)

the form factor (1.2) appearing in the last two terms of the stress-strain matrix accounts for the fact that the transverse shearing stresses produce too little strain energy [2].

\([B_A]\), and \([B_B]\) are matrices containing the shape functions and the associated derivatives, and consist of the sub-matrices

\[
[B_{Ai}] = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-b_i & -N_i & 0 \\
-a_i & 0 & N_i
\end{bmatrix}
\quad \quad
[B_{Bi}] = \begin{bmatrix}
0 & 0 & a_i \\
0 & -b_i & 0 \\
0 & -a_i & b_i \\
0 & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (8.34)

with \(a_i\), and \(b_i\) defined as

\[
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix} = \begin{bmatrix}
\xi_{ix} & \eta_{ix} \\
\xi_{iy} & \eta_{iy}
\end{bmatrix}\begin{bmatrix}
N_{i,x} \\
N_{i,y}
\end{bmatrix}
\]  \hspace{1cm} (8.35)

The integral indicated in equation 8.32 is evaluated numerically using a standard Gaussian quadrature with the following points and weights:

<table>
<thead>
<tr>
<th>(\xi)</th>
<th>(\eta)</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.774596669241483</td>
<td>-0.774596669241483</td>
<td>25/81</td>
</tr>
</tbody>
</table>
Mass Matrix

The element mass matrix is evaluated from the integral

\[
[M] = \rho \int_A \left( 2[N_A]^T[N_A] + \frac{h^2}{6}[N_B]^T[N_B] \right) dA
\]

with \([N_A]\), and \([N_B]\) being the following shape function matrices,

\[
[N_A] = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix} \quad \quad [N_B] = \begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The integral indicated in equation 8.36 is evaluated numerically using a standard Gaussian quadrature with the points and weights shown in the Stiffness Matrix section.

Damping Matrix

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

where a and b are user defined constants.
**3-D General Shell**

The element is of rectangular shape defined by eight geometric nodes and is arbitrarily oriented in the 3-D space. Each node has three translational degrees of freedom in the global coordinate system, and two rotational degrees of freedom in the plane tangent on the node under consideration.

---

**Shape Functions**

The shape functions of the 8-node general shell element are

\[
N_i = \frac{1}{4} (1 + \xi_o)(1 + \eta_i)(-1 + \xi_o + \eta_i) \quad i = 1,2,3,4 \\
N_i = \frac{1}{2} (1 - \xi^2)(1 + \eta_i) \quad i = 5,7 \\
N_i = \frac{1}{2} (1 + \xi_o)(1 - \eta^2) \quad i = 6,8
\]

where $\xi_o = \xi \xi_i$ and $\eta_o = \eta \eta_i$. 

8.39
Stiffness Matrix

The element stiffness matrix is given as

\[
[K] = \int_A \left( 2[B_A]^T [E][B_A] + \frac{2}{3}[B_B]^T [E][B_B] \right) dA
\]

where:

\[ [E] \] is the stress-strain matrix

\[
[E] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix}
1-v & \nu & 1-2v \\
\nu & 1-2v & \frac{1-2v}{2} \\
1-2v & \frac{1-2v}{2} & \frac{1-2v}{2(1.2)}
\end{bmatrix}
\]

the form factor (1.2) appearing in the last two terms of the stress-strain matrix accounts for the fact that the transverse shearing stresses produce too little strain energy [2].

\[ [B_A] \] and \[ [B_B] \] are matrices containing the shape functions and the associated derivatives,

\[
[B_A] = \begin{bmatrix}
a_i & 0 & 0 & -d_i l_{2i} & d_i l_{ii} \\
0 & b_i & 0 & -e_i m_{2i} & e_i m_{ii} \\
0 & 0 & c_i & -g_i n_{2i} & g_i n_{ii} \\
b_i & a_i & 0 & -e_i l_{2i} - d_i m_{2i} & e_i l_{ii} + d_i m_{ii} \\
0 & c_i & b_i & -g_i m_{2i} - e_i n_{2i} & g_i m_{ii} + e_i n_{ii} \\
c_i & 0 & a_i & -d_i n_{2i} - g_i l_{2i} & d_i n_{ii} + g_i l_{ii}
\end{bmatrix}
\]

\[ [B_B] = \frac{h}{2} [T] \]

\[
[B_B] = \begin{bmatrix}
0 & 0 & 0 & -a_i l_{2i} & a_i l_{ii} \\
0 & 0 & 0 & -b_i m_{2i} & b_i m_{ii} \\
0 & 0 & 0 & -c_i n_{2i} & c_i n_{ii} \\
0 & 0 & 0 & -b_i l_{2i} - a_i m_{2i} & b_i l_{ii} + a_i m_{ii} \\
0 & 0 & 0 & -c_i m_{2i} - b_i n_{2i} & c_i m_{ii} + b_i n_{ii} \\
0 & 0 & 0 & -a_i n_{2i} - c_i l_{2i} & a_i n_{ii} + c_i l_{ii}
\end{bmatrix}
\]

The parameters \( a_i \) through \( g_i \) are defined as
Matrix \([T]\) is the transformation matrix containing the directional cosines of the normal and tangential vectors at the integration point, and \(l_{ji}, m_{ji}, n_{ji}, j=1,2\) represent the directional cosines of the two tangential vectors at node \(i\). The integral indicated in equation 8.40 is evaluated numerically using a standard Gaussian quadrature with the points and weights listed in the element \(3-D\) Horizontal Plate - Bending Only.

**Mass Matrix**

The element mass matrix is evaluated from the integral

\[
[M] = \rho \int \int \left( 2[N_A]^T[N_A] + \frac{2}{3}[N_B]^T[N_B] \right) dA
\]

with \([N_A]\), and \([N_B]\) being the following shape function matrices,

\[
[N_{Ai}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} N_i \\
[N_{Bi}] = \begin{bmatrix} 0 & 0 & 0 & -l_{2i} & l_{1i} \\ 0 & 0 & 0 & -m_{2i} & m_{1i} \\ 0 & 0 & 0 & -n_{2i} & n_{1i} \end{bmatrix} N_i
\]

The integral indicated in equation 8.45 is evaluated numerically using a standard Gaussian quadrature with the points and weights listed in the element \(3-D\) Horizontal Plate - Bending Only.

**Damping Matrix**

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

where \(a\) and \(b\) are user defined constants.

**2-D Constant Strain Triangle**
This is a triangular element defined by three geometric nodes. Each node has two translational degrees of freedom in the global coordinate system. It is used to model two dimensional plane strain solids and soils.

**Shape Functions**
The shape functions of the 3-node constant strain triangle are

\[ N_i = L_i \quad i = 1,2,3 \]

where \( L_i \) are the area coordinates.

**Stiffness Matrix**
The element stiffness matrix is given as

\[ [K] = \int_A [B]^T [D] [B] dA \]

where \([D]\) is the elasticity matrix for a plane strain isotropic material expressed as,

\[ [D] = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix} 1 & \frac{v}{1-v} & 0 \\ \frac{v}{1-v} & 1 & 0 \\ 0 & 0 & \frac{(1-2v)}{2(1-v)} \end{bmatrix} \]

and \([B]\) consists of the sub-matrices \([B]_i, i=1,2,3\), defined as
The integral indicated in equation 8.49 is evaluated numerically using a standard Gaussian quadrature with one integration point at \( L_1 = L_2 = L_3 = 1/3 \) and corresponding weight \( w = 1.0 \).

### Mass Matrix
The element mass matrix is evaluated on the basis of the consistent formulation as

\[
[M] = \rho \int [N]^T [N] dA \tag{8.52}
\]

where \( \rho \) is the mass density of the elastic medium, and the integral is evaluated numerically, similarly to the stiffness matrix.

### Damping Matrix
For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C] \tag{8.53}
\]

where \( a \) and \( b \) are user defined constants.
2-D  Linear Strain Triangle

This is a triangular element defined by six geometric nodes. Each node has two translational degrees of freedom in the global coordinate system. It is used to model two dimensional plane strain solids and soils.

Shape Functions
The shape functions of the 6-node linear strain triangle are

\[ N_i = L_i \left(2L_i - 1\right) \quad i = 1,2,3 \]
\[ N_i = 4L_p L_q \quad i = 4,5,6, \quad p = i - 3, \quad q = i - 2 \]

where \( L_i \) are the area coordinates.

Stiffness Matrix
The element stiffness matrix is given as

\[ [K] = \int [B]^T [D] [B] dA \]

where \([D]\) is the elasticity matrix for a plane strain isotropic material defined in equation 8.50, and \([B]\) consists of the sub-matrices \([B]_i\), \(i=1,2,...,6\) defined in equation 8.51. The integral indicated in equation 8.55 is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.6.
Table 8.6  Integration Points and Weights

<table>
<thead>
<tr>
<th>L_1</th>
<th>L_2</th>
<th>L_3</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.0</td>
<td>1/3</td>
</tr>
<tr>
<td>0.0</td>
<td>0.5</td>
<td>0.5</td>
<td>1/3</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Mass Matrix
The element mass matrix is evaluated on the basis of the consistent formulation as

\[
[M] = \rho \int_A [N]^T[N]dA \tag{8.56}
\]

where \(\rho\) is the mass density of the elastic medium, and the integral is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.6.

Damping Matrix
For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C] \tag{8.57}
\]

where \(a\) and \(b\) are user defined constants.
**2-D Constant Strain Quadrilateral**

This is a triangular element defined by four geometric nodes. Each node has two translational degrees of freedom in the global coordinate system. It is used to model two dimensional plane strain solids and soils.

**Shape Functions**

The shape functions of the 4-node constant strain triangle are

\[
N_1 = \frac{1}{4} (1 - \xi)(1 - \eta) \quad N_2 = \frac{1}{4} (1 + \xi)(1 - \eta) \\
N_3 = \frac{1}{4} (1 + \xi)(1 + \eta) \quad N_4 = \frac{1}{4} (1 - \xi)(1 + \eta)
\]

**Stiffness Matrix**

The element stiffness matrix is given as

\[
[K] = \int_{A} [B]^T[D][B]dA
\]
where $[D]$ is the elasticity matrix for a plane strain isotropic material defined in equation 8.50, and $[B]$ consists of the sub-matrices $[B]_i, i=1,2,3,4$ defined in equation 8.51. The integral indicated in equation 8.59 is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.7.

Table 8.7 Integration Points and Weights

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\eta$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.577350269189626</td>
<td>-0.577350269189626</td>
<td>1.0</td>
</tr>
<tr>
<td>0.577350269189626</td>
<td>-0.577350269189626</td>
<td>1.0</td>
</tr>
<tr>
<td>0.577350269189626</td>
<td>0.577350269189626</td>
<td>1.0</td>
</tr>
<tr>
<td>-0.577350269189626</td>
<td>0.577350269189626</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**Mass Matrix**

The element mass matrix is evaluated on the basis of the consistent formulation as

$$[M] = \rho \int_A [N]^T [N] dA$$  \hspace{1cm} 8.60

where $\rho$ is the mass density of the elastic medium, and the integral is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.7.

**Damping Matrix**

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

$$[C] = a[M] + b[C]$$  \hspace{1cm} 8.61

where $a$ and $b$ are user defined constants.
2-D Linear Strain Quadrilateral

This is a triangular element defined by eight geometric nodes. Each node has two translational degrees of freedom in the global coordinate system. It is used to model two dimensional plane strain solids and soils.

Shape Functions

The shape functions of the 4-node constant strain triangle are

\[ N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-1 - \xi - \eta) \]
\[ N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(-1 + \xi - \eta) \]
\[ N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(-1 + \xi + \eta) \]
\[ N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-1 - \xi + \eta) \]
\[ N_5 = \frac{1}{2}(1 - \eta)(1 - \xi^2) \]
\[ N_6 = \frac{1}{2}(1 + \xi)(1 - \eta^2) \]
\[ N_7 = \frac{1}{2}(1 + \eta)(1 - \xi^2) \]
\[ N_8 = \frac{1}{2}(1 - \xi)(1 - \eta^2) \]
Stiffness Matrix
The element stiffness matrix is given as

\[
[K] = \int_A [B]^T [D][B] dA \tag{8.63}
\]

where \([D]\) is the elasticity matrix for a plane strain isotropic material defined in equation 8.50, and \([B]\) consists of the sub-matrices \([B]_i, i=1,2,...,8\) defined in equation 8.51. The integral indicated in equation 8.62 is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.8.

**Table 8.8 Integration Points and Weights**

<table>
<thead>
<tr>
<th>ξ</th>
<th>η</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.774596669241483</td>
<td>-0.774596669241483</td>
<td>1/3.24</td>
</tr>
<tr>
<td>0.0</td>
<td>-0.774596669241483</td>
<td>1/2.025</td>
</tr>
<tr>
<td>0.774596669241483</td>
<td>-0.774596669241483</td>
<td>1/3.24</td>
</tr>
<tr>
<td>-0.774596669241483</td>
<td>0.0</td>
<td>1/2.025</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>1/1.265625</td>
</tr>
<tr>
<td>0.774596669241483</td>
<td>0.0</td>
<td>1/2.025</td>
</tr>
<tr>
<td>-0.774596669241483</td>
<td>0.774596669241483</td>
<td>1/3.24</td>
</tr>
<tr>
<td>0.0</td>
<td>0.774596669241483</td>
<td>1/2.025</td>
</tr>
<tr>
<td>0.774596669241483</td>
<td>0.774596669241483</td>
<td>1/3.24</td>
</tr>
</tbody>
</table>

Mass Matrix
The element mass matrix is evaluated on the basis of the consistent formulation as

\[
[M] = \rho \int_A [N]^T [N] dA \tag{8.64}
\]

where \(\rho\) is the mass density of the elastic medium, and the integral is evaluated numerically.
Damping Matrix

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

8.65

where \( a \) and \( b \) are user defined constants.
**3-D 10 Node Tetrahedron**

This is a solid element defined by ten geometric nodes. Each node has three translational degrees of freedom in the global coordinate system. It is used to model three dimensional solids and soils.

**Shape Functions**

The shape functions of the 3-node constant strain triangle are

\[
N_i = L_i \left(2L_i - 1\right) \quad i = 1,2,3,4
\]

\[
N_5 = 4L_1L_2 \quad N_6 = 4L_2L_3 \quad N_7 = 4L_3L_4
\]

\[
N_8 = 4L_1L_4 \quad N_9 = 4L_2L_4 \quad N_{10} = 4L_3L_4
\]

where \(L_i\) are the volume coordinates

**Stiffness Matrix**

The element stiffness matrix is given as

\[
[K] = \int_{A} [B]^T [D][B] \, dA
\]

where \([D]\) is the elasticity matrix for an isotropic material expressed as,
and \([B]\) consists of the sub-matrices \([B]_i, i=1,2,...,10\) defined as

\[
[B]_i = \begin{bmatrix}
\frac{\partial N_i}{\partial x} & 0 & 0 \\
0 & \frac{\partial N_i}{\partial y} & 0 \\
0 & 0 & \frac{\partial N_i}{\partial z} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\
0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x}
\end{bmatrix}
\]

The integral indicated in equation 8.67 is evaluated numerically using a standard Gaussian quadrature with the integration points and weights shown in Table 8.9.

<table>
<thead>
<tr>
<th>L_1</th>
<th>L_2</th>
<th>L_3</th>
<th>L_4</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>¼</td>
<td>1/4</td>
<td>1/4</td>
<td>¼</td>
<td>-4/5</td>
</tr>
<tr>
<td>½</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>9/20</td>
</tr>
<tr>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>9/20</td>
</tr>
<tr>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>½</td>
<td>9/20</td>
</tr>
</tbody>
</table>

Mass Matrix
The element mass matrix is evaluated on the basis of the consistent formulation as
\[
[M] = \rho \int_{A} [N]^{T}[N]dA
\]

where \( \rho \) is the mass density of the elastic medium, and the integral is also evaluated numerically using a standard Gaussian quadrature with the integration points and weights shown in Table 8.9.

**Damping Matrix**

For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\]

where \( a \) and \( b \) are user defined constants.
This is a solid element defined by 15 geometric nodes. Each node has three translational degrees of freedom in the global coordinate system. It is used to model three dimensional solids and soils.

Shape Functions
The shape functions of the 15-node pentahedron are

\[
N_i = \frac{1}{2} \left( L_i \left( 2L_i - 1 \right) \left( 1 - \xi \right) - L_i \left( 1 - \xi^2 \right) \right) \quad i = 1, 2, 3
\]
\[
N_i = \frac{1}{2} \left( L_j \left( 2L_j - 1 \right) \left( 1 + \xi \right) - L_j \left( 1 - \xi^2 \right) \right) \quad i = 4, 5, 6 \quad p = 1, 2, 3
\]
\[
N_i = 2L_p L_q \left( 1 - \xi \right) \quad i = 7, 8, 9 \quad p = 1, 2, 3 \quad q = 2, 3, 1
\]
\[
N_i = 2L_p L_q \left( 1 + \xi \right) \quad i = 10, 11, 12 \quad p = 1, 2, 3 \quad q = 2, 3, 1
\]
\[
N_i = 2L_p \left( 1 - \xi^2 \right) \quad i = 13, 14, 15 \quad p = 1, 2, 3
\]
where \( L_i \) are the area coordinates at level \( \xi \), \(-1 \leq \xi \leq 1\)

**Stiffness Matrix**
The element stiffness matrix is given as

\[
[K] = \int_{A} [B]^T [D] [B] dA
\] 8.73

where \([D]\) is the elasticity matrix for a plane strain isotropic material defined in equation 8.68, and \([B]\) consists of the sub-matrices \([B]_i\), \(i=1,2,...,15\) defined in equation 8.69. The integral indicated in equation 8.73 is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.10

**Table 8.10  Integration Points and Weights**

<table>
<thead>
<tr>
<th>( L_1 )</th>
<th>( L_2 )</th>
<th>( L_3 )</th>
<th>( \xi )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.0</td>
<td>-0.577350269189626</td>
<td>1.0/3.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
<td>-0.577350269189626</td>
<td>1.0/3.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.5</td>
<td>0.5</td>
<td>-0.577350269189626</td>
<td>1.0/3.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.0</td>
<td>0.577350269189626</td>
<td>1.0/3.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
<td>0.577350269189626</td>
<td>1.0/3.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.5</td>
<td>0.5</td>
<td>0.577350269189626</td>
<td>1.0/3.0</td>
</tr>
</tbody>
</table>

**Mass Matrix**
The element mass matrix is evaluated on the basis of the consistent formulation as

\[
[M] = \rho \int_{A} [N]^T [N] dA
\] 8.74

where \( \rho \) is the mass density of the elastic medium, and the integral is evaluated numerically, as in the stiffness matrix.

**Damping Matrix**
For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[
[C] = a[M] + b[C]
\] 8.75

where \( a \) and \( b \) are user defined constants.
3-D 20 Node Hexahedron

This is a solid element defined by 20 geometric nodes. Each node has three translational degrees of freedom in the global coordinate system. It is used to model three dimensional solids and soils.

Shape Functions

The shape functions of the 20-node hexahedron are

\[
N_i = \begin{align*}
N_i &= \frac{1}{8} (1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta)(\xi_i \xi + \eta_i \eta + \zeta_i \zeta - 2) & i = 1,2,...,8 \\
N_i &= \frac{1}{4} (1 - \xi_i^2)(1 + \eta_i \eta)(1 + \zeta_i \zeta) & i = 9,11,13,15 \\
N_i &= \frac{1}{4} (1 + \xi_i \xi)(1 + \eta_i^2)(1 + \zeta_i \zeta) & i = 10,12,14,16 \\
N_i &= \frac{1}{4} (1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i^2) & i = 17,18,19,20
\end{align*}
\]

where \(\xi_i, \eta_i, \zeta_i\) are the coordinates of the element node with respect to the local coordinate system, \(-1 \leq \xi \leq 1, -1 \leq \eta \leq 1, -1 \leq \zeta \leq 1\).
Stiffness Matrix

The element stiffness matrix is given as

\[
[K] = \int_{\Omega} [B]^T [D] [B] d\Omega
\]

where \([D]\) is the elasticity matrix for a plane strain isotropic material defined in equation 8.68, and \([B]\) consists of the sub-matrices \([B]_i, i=1,2,...,20\) defined in equation 8.69. The integral indicated in equation 8.77 is evaluated numerically using a standard Gaussian quadrature with the integration points and weights defined in Table 8.11

<table>
<thead>
<tr>
<th>ξ</th>
<th>η</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.77459669241483</td>
<td>0.77459669241483</td>
</tr>
<tr>
<td>0.0</td>
<td>-0.77459669241483</td>
<td>-0.77459669241483</td>
</tr>
<tr>
<td>-0.77459669241483</td>
<td>0.77459669241483</td>
<td>0.77459669241483</td>
</tr>
<tr>
<td>0.77459669241483</td>
<td>0.77459669241483</td>
<td>0.77459669241483</td>
</tr>
</tbody>
</table>

W0 = 0.171467763 W1 = 0.274348421
W2 = 0.438957474 W3 = 0.702331959
Mass Matrix
The element mass matrix is evaluated on the basis of the consistent formulation as

\[ [M] = \rho \int_{A} [N]^T[N]dA \]  

where \( \rho \) is the mass density of the elastic medium, and the integral is evaluated numerically, similarly to the stiffness matrix.

Damping Matrix
For time domain analysis the damping matrix is evaluated according to the proportional damping formulation (Rayleigh damping) as a linear combination of the stiffness and mass matrices,

\[ [C] = a[M] + b[C] \]  

where \( a \) and \( b \) are user defined constants.
2-D Vertical Transmitting Boundaries

The two dimensional vertical transmitting boundary is developed for frequency domain analysis. This elements are defined by all nodes bounding each side of the layered soil medium. They are used to model the far field motion of the infinite half space. These elements are attached directly to the finite elements of the soil region.

The element formulation is based on the calculation by the finite element method of the modes of vibration satisfying the boundary conditions on the surface and the bedrock. Subsequently, the solution in the far field is written as a linear combination of these semi-discrete modes. Details of this formulation can be found in the reference by J. Tassoulas and E. Kausel [8].
3-D Vertical Transmitting Boundaries

The two dimensional vertical transmitting boundary is developed for frequency domain analysis. These elements are defined by all nodes on the surface of the soil cylinder. They are used to model the far field motion of the 3-D infinite half space. These elements are attached directly to the finite elements of the soil region.

The three dimensional hyperelement is a direct extension of the two dimensional one introduced in the previous section. The development of this element by Lin and Tassoulas [3] who extended the axisymmetric boundaries of Waas [4] and Kausel [5] provides the basis for the hyperelement implemented in FETA. The approximate treatment of the half space underlying a layered stratum for both 2-D and 3-D is due to Hull and Kausel [6].
2-D Absorbing Boundary

The two dimensional absorbing boundary is developed for time or frequency domain analysis. This element is defined on the boundaries of the layered soil medium and consist of two or three nodes, depending on the soil elements that they are attached to. They are used to model the far field motion of the infinite half space by introducing appropriate dashpots. These elements are assembled directly to the damping matrix of the system under consideration.

Damping Matrix

The absorbing boundary consists of vertical and horizontal dashpots per unit length of the boundary. The associated damping coefficients are proportional to the pressure, $V_1$, and shear, $V_2$, wave velocities and are evaluated as

$$C_V = \rho V_1$$

$$C_H = \rho V_2$$

The damping matrix is evaluated according to the consistent formulation.
3-D Absorbing Boundary

The three dimensional absorbing boundary is developed for time or frequency domain analysis. This elements are defined on the boundary of the layered soil medium and consist of six or eight nodes, depending on the soil elements they are attached to. They are used to model the far field motion of the infinite half space by introducing appropriate dashpots. These elements are assembled directly to the damping matrix of the system under consideration.

Damping Matrix

The absorbing boundary consists of one vertical and two horizontal dashpots per unit length of the boundary. The associated damping coefficients are proportional to the pressure, $V_1$, and shear, $V_2$, wave velocities and are evaluated as

$$C_V = \rho V_1$$
$$C_H = \rho V_2$$

The damping matrix is evaluated according to the consistent formulation.
2-D Rigid Link

The two dimensional rigid link is defined by two geometric nodes. This element implements the constrained equations for the motion of the two nodes in all corresponding degrees of freedom. Each node has two sets of two translational and one rotational degrees of freedom in the global coordinate system. The first set corresponds to displacements, and the second one corresponds to forces. In addition, a reference node, R, is assigned the three degrees of freedom shown in the figure to implement the coupling of the motion of the two geometric nodes. This element can be used by itself when both nodes have all three degrees of freedom, or combined with a massless 2-D beam element, that provides the appropriate stiffness, in any other case.

Constrained matrix

The constrained matrix is formulated on the basis of the Lagrange multipliers, and for the 2-D rigid link element is given as:
$[L] = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
symm & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
R_1 x \\
R_1 y \\
R_2 x \\
R_2 y \\
symm \\
\end{bmatrix} = \begin{bmatrix}
8.82 
\end{bmatrix}$

where $R_{1x} = X_R - X_1$, $R_{1y} = Y_R - Y_1$, $R_{2x} = X_R - X_2$, and $R_{2y} = Y_R - Y_2$. 
3-D Rigid Link

The three dimensional rigid link is defined by two geometric nodes. This element implements the constrained equations for the motion of the two nodes in all corresponding degrees of freedom. Each node has two sets of three translational and three rotational degrees of freedom in the global coordinate system. The first set corresponds to displacements, and the second one corresponds to forces. In addition, a reference node, R, is assigned the six degrees of freedom shown in the figure to implement the coupling of the motion of the two geometric nodes. This element can be used by itself when both nodes have all six degrees of freedom, or combined with a massless 3-D beam element, that provides the required stiffness, in any other case.
Constrained matrix

The constrained matrix is formulated on the basis of the Lagrange multipliers, and for the 3-D rigid link element is given as:

\[
[L] = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & L_1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & L_2 \\
0 & L_1^T & 0 & L_2^T & 0 & 0
\end{bmatrix}
\]

where

\[
[L_1] = \begin{bmatrix}
1 & 0 & 0 & 0 & R_1 & -R_1 \\
1 & 0 & -R_1 & 0 & R_1 & 0 \\
1 & R_1 & -R_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
[L_2] = \begin{bmatrix}
1 & 0 & 0 & 0 & R_2 & -R_2 \\
1 & 0 & -R_2 & 0 & R_2 & 0 \\
1 & R_2 & -R_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
2-D Rigid Footing

The 2-D rigid footing consists of all geometric nodes in contact with the user defined line which represents the rigid region, and an additional reference node that enforces the rigid conditions. Each node in the rigid footing along with the common reference node becomes a 2-D rigid footing element and its formulation is based on the Lagrange multipliers. Each node has two sets of two translational degrees of freedom in the global coordinate system. The first set corresponds to displacements, and the second one corresponds to forces. The reference node has two translational and one rotational degrees of freedom corresponding to the translations and rotation of the footing as rigid body.

Constrained matrix

The constrained matrix is formulated on the basis of the Lagrange multipliers, and for the 2-D rigid footing element is given as:

\[
[L] = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & R_y \\
0 & -1 & 0 & 0 & 0 & 1 & -R_x \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & R_y & -R_x & 0 & 0 & 0 \\
0 & 0 & R_y & -R_x & 0 & 0 & 0
\end{bmatrix}
\]

where \( R_x = X_R - X_i \), \( R_y = Y_R - Y_i \)
3-D Rigid Footing

The 3-D rigid footing consists of all geometric nodes in contact with the user defined surface which represents the rigid region, and an additional reference node that enforces the rigid conditions. Each node in the rigid footing along with the common reference node becomes a 3-D rigid footing element and its formulation is based on the Lagrange multipliers. Each node has two sets of three translational degrees of freedom in the global coordinate system. The first set corresponds to displacements, and the second one corresponds to forces. The reference node has three translational and three rotational degrees of freedom corresponding to the translations and rotations of the footing as rigid body.

Constrained matrix

The constrained matrix is formulated on the basis of the Lagrange multipliers, and for the 3-D rigid footing element is given as:

\[
[L] = \begin{bmatrix}
0 & -I & 0 & 0 \\
-I & 0 & I & R \\
0 & I & 0 & 0 \\
0 & [R]^T & 0 & 0 \\
\end{bmatrix}
\]

8.86
where $[I]$ is the identity matrix and

$$[R] = \begin{bmatrix}
0 & R_z & -R_y \\
-R_z & 0 & R_x \\
R_y & -R_x & 0
\end{bmatrix}$$

with

$$R_x = X_R - X_n$$

$$R_y = Y_R - Y_n$$

$$R_z = Z_R - Z_n$$

8.87
2-D Frictional Contact Element

The 2-D frictional contact element is an interface element between two bodies in contact. It consists of three pairs of nodes. The two nodes of each pair are initially coincident. Each node has two degrees of freedom in the local coordinate system, i.e. the tangential velocity and the normal displacement. In addition, three parameters are required to completely describe the element, i.e. the coefficient of friction $\mu$, and the penalty coefficients $k_n$, and $k_s$ that represent the relative stiffness in the normal and tangential direction, respectively. This element is used in time domain analysis only.

Formulation Parameters

The sides of the element that may be in frictional contact are defined by nodes 1-3-5 and 2-4-6. Let $x^{(i)}$, $u^{(i)}$, $v^{(i)}$ be the position, displacement and velocity vectors of node $i$. At any point $s$ along the element, we can compute

\[
x = x^{(1)} N_1(s) + x^{(3)} N_2(s) + x^{(5)} N_3(s)
\]

\[
u^+ = u^{(2)} N_1(s) + u^{(4)} N_2(s) + u^{(6)} N_3(s)
\]

\[
u^- = u^{(1)} N_1(s) + u^{(3)} N_2(s) + u^{(5)} N_3(s)
\]
\[ \mathbf{v}^+ = \mathbf{v}^{(2)}_1 \mathbf{N}_1(s) + \mathbf{v}^{(4)}_2 \mathbf{N}_2(s) + \mathbf{v}^{(6)}_3 \mathbf{N}_3(s) \]
\[ \mathbf{v}^- = \mathbf{v}^{(1)}_1 \mathbf{N}_1(s) + \mathbf{v}^{(3)}_2 \mathbf{N}_2(s) + \mathbf{v}^{(5)}_3 \mathbf{N}_3(s) \]

where \( \mathbf{N}_i \) are the quadratic Lagrangian interpolation functions. In terms of the above we can determine the unit vectors, tangent and normal to the element:

\[
\mathbf{s} = \begin{bmatrix} s_x \\ s_y \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} -s_y \\ s_x \end{bmatrix}
\]

The separation and relative tangential velocity are then given by:

\[
\delta_n = \left( \mathbf{u}^+ - \mathbf{u}^- \right)^\mathbf{T} \mathbf{n}
\]
\[
\mathbf{v}_s = \left( \mathbf{v}^+ - \mathbf{v}^- \right)^\mathbf{T} \mathbf{s}
\]

where negative \( \delta_n \) indicates penetration. Finally, the forces on the interface nodes are computed as:

If \( \delta_n \leq \epsilon \)

\[
\mathbf{F}_n = k_n \delta_n
\]
\[
\mathbf{F}_s = \begin{cases} \mu \mathbf{F}_n \\ k_s \mathbf{v}_s \\ -\mu \mathbf{F}_n \end{cases} \quad \text{if} \quad \frac{\mu \mathbf{F}_n}{k_s} \leq \mathbf{v}_s \leq \frac{\mu \mathbf{F}_n}{k_s}
\]

If \( \delta_n > \epsilon \)

\[
\mathbf{F}_n = 0
\]
\[
\mathbf{F}_s = 0
\]

where \( k_n \) and \( k_s \) are the penalty parameters indicating the relative stiffness between the two bodies and \( \epsilon \) is a positive tolerance.
Stiffness Matrix
The contribution to the stiffness matrix is evaluated as

$$[K] = \int [N]^T[D_n][Q_n]^T[N]ds$$  \hspace{1cm} 8.91

where

$$[N] = \left[ \begin{array}{c} -N_1 \\ N_1 \\ -N_2 \\ N_2 \\ -N_3 \\ N_3 \end{array} \right], \quad [N_i] = \left[ \begin{array}{c} N_i \\ 0 \\ 0 \end{array} \right]$$  \hspace{1cm} 8.92

are the shape functions, $Q_n$ contains the directional cosines of the normal and

$$D_n = \begin{cases} k_n & \text{if } \delta_n \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} 8.93

Damping Matrix
The contribution of the element in the damping matrix of the system is calculated as

$$[C] = \int [N]^T[D_s][Q_s]^T[N]ds$$  \hspace{1cm} 8.94

where $[N]$ are the shape functions as in Equation 8.02, $Q_s$ contains the directional cosines of the tangent vector and

$$D_s = \begin{cases} k_s & \text{if } \delta_n \leq \varepsilon \text{ and } |v_s| \leq \frac{|\mu F_n|}{k_s} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} 8.95

Contribution to the Right Hand Side
The contribution to the right hand side of the system of equations is calculated as

$$\{R\} = -\int [N]^T[Q][F]ds$$  \hspace{1cm} 8.96

where $[N]$ are the shape functions as indicated in Equation 8.02, $[Q]$ contains the directional cosines and $\{F\}$ is the vector of the forces on the frictional interface as calculated in equation 8.91.
28 2-D Fluid Element

The 2-D fluid element is developed on the basis of the Helmholtz’s or Laplace’s equation for the time harmonic motion of compressible and incompressible fluids, respectively. It is a 2-D 3-node or 4-node surface patch element and the degrees of freedom correspond to the fluid velocity potential. Along with the fluid-solid interface is used to model fluid domains in a fluid-structure-interaction applications.

Formulation

The governing equation for time harmonic motion of a compressible fluid is the well known Helmholtz’s equation:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\omega^2}{C^2} \phi = 0
\]

8.97

while the Laplace’s equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]

8.98

must be satisfied in an incompressible fluid. In equations 8.97 and 8.98, \( \phi \) denotes the fluid velocity potential and \( C \) is the pressure wave velocity.

Applying the variational approach, both sides of equation 8.97 are multiplied by \( \rho \delta \phi \) and integrated by parts over the domain \( \Omega \). This approach yields:
\begin{align*}
-\rho \int_{\Omega} \left[ \frac{\partial}{\partial x} \left( \delta \phi \right) \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y} \left( \delta \phi \right) \frac{\partial \phi}{\partial y} \right] d\Omega &+ \frac{\omega^2}{C^2} \rho \int_{\Omega} (\delta \phi) \phi d\Omega = \\
-\rho \int_{\Omega} \left[ \frac{\partial}{\partial x} \left( \delta \phi \right) \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y} \left( \delta \phi \right) \frac{\partial \phi}{\partial y} \right] d\Omega &+ \frac{\omega^2}{C^2} \rho \int_{\Omega} (\delta \phi) \phi d\Omega = \\
8.99
\end{align*}

Applying Green’s theorem on the right hand side of equation 8.99 in order to change the integral over the domain \( \Omega \) to the integral over the domain \( B \) yields:

\begin{align*}
-\rho \int_{\Omega} \left[ \frac{\partial}{\partial x} \left( \delta \phi \right) \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y} \left( \delta \phi \right) \frac{\partial \phi}{\partial y} \right] d\Omega &+ \frac{\omega^2}{C^2} \rho \int_{\Omega} (\delta \phi) \phi d\Omega = \\
-\rho \int_{B} \delta \phi \left( \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y \right) dB
\end{align*}

Considering triangular, or quadrilateral isoparametric elements with linear interpolation of the amplitude of the velocity potential equation 8.100 can be written as:

\begin{align*}
\left[ -\rho \int_{\Omega} \left( \frac{\partial N_i}{\partial x} \left( \frac{\partial N_i}{\partial x} \right)^T + \frac{\partial N_i}{\partial y} \left( \frac{\partial N_i}{\partial y} \right)^T \right] d\Omega + \frac{\omega^2}{C^2} \rho \int_{\Omega} NN^T d\Omega \right] \phi_i
&= \left\{ R_i \right\} \\
8.101
\end{align*}

where \( N = N_i, i=1,.. \) number of element nodes, is the vector of the shape functions associated to the element that is used (see equations 8.48, 8.58), \( \phi_i \) is the velocity potential at the nodes of the element, and

\begin{equation}
R_i = -\rho \int_{B} N_i \left( \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y \right) dB \tag{8.102}
\end{equation}

represents the amplitude of the consistent mass flux into the domain at node \( i \). In the case of an incompressible fluid, the second integral in equation 8.1015 will drop out.
**2-D Fluid-Solid Interface Element**

The 2-D fluid-solid interface element is an element that relates the nodal velocity potentials of the fluid region to the nodal horizontal and vertical displacements of the solid domain. This element is created automatically by FETA at the interface between fluid and solid domains and is invisible to the user.

---

Formulation

At the interface between the fluid and solid finite elements, the nodal velocity potentials of fluid must be related to the nodal horizontal and vertical displacements of the solid finite element. This can be done by taking the contributions of the consistent mass flux, and consistent nodal forces on the common boundary S to the left-hand side of the equations and for convenience storing the result as a matrix of an auxiliary interface element to be assembled with the matrices corresponding to the solid and fluid finite elements.

The contribution of the mass flux through S corresponding to the starting (1) and ending (2) nodes of the interface element are given explicitly after integration in terms of the outward normal and the horizontal and vertical displacements U and W as
On the other hand, the contribution of the consistent horizontal and vertical nodal forces $H$ and $V$ from $S$ to the starting (1) and ending (2) nodes of the interface element are given explicitly after integration in terms of the fluid pressure on $S$ as:

$$R_1^s = -i\omega \rho L \begin{bmatrix} n_x \frac{1}{3} & n_y \frac{1}{3} & n_x \frac{1}{6} & n_y \frac{1}{6} \end{bmatrix} \begin{bmatrix} U_1 \\ W_1 \\ U_2 \\ W_2 \end{bmatrix}$$

$$R_2^s = -i\omega \rho L \begin{bmatrix} n_x \frac{1}{6} & n_y \frac{1}{6} & n_x \frac{1}{3} & n_y \frac{1}{3} \end{bmatrix} \begin{bmatrix} U_1 \\ W_1 \\ U_2 \\ W_2 \end{bmatrix}$$

On the other hand, the contribution of the consistent horizontal and vertical nodal forces $H$ and $V$ from $S$ to the starting (1) and ending (2) nodes of the interface element are given explicitly after integration in terms of the fluid pressure on $S$ as:

$$H_i^s = -i\omega \rho L \begin{bmatrix} n_x \frac{1}{3} & n_x \frac{1}{3} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$V_i^s = -i\omega \rho L \begin{bmatrix} n_y \frac{1}{3} & n_y \frac{1}{3} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$H_2^s = -i\omega \rho L \begin{bmatrix} n_x \frac{1}{6} & n_x \frac{1}{3} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$V_2^s = -i\omega \rho L \begin{bmatrix} n_y \frac{1}{6} & n_y \frac{1}{3} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

The relations 8.103 and 8.104 represent the contributions of the consistent mass flux and consistent nodal forces on the interface $S$. If these contributions are taken to the left-hand side of the equations, the matrix of the auxiliary element is formed as:

$$Q = i\omega \rho L \begin{bmatrix} 0 & 0 & n_x \frac{1}{3} & n_x \frac{1}{3} & n_x \frac{1}{6} & n_x \frac{1}{6} \\ 0 & 0 & n_y \frac{1}{3} & n_y \frac{1}{3} & n_y \frac{1}{6} & n_y \frac{1}{6} \\ n_x \frac{1}{6} & n_x \frac{1}{3} & 0 & 0 & 0 & 0 \\ n_y \frac{1}{6} & n_y \frac{1}{3} & 0 & 0 & 0 & 0 \\ n_x \frac{1}{3} & n_x \frac{1}{6} & 0 & 0 & 0 & 0 \\ n_y \frac{1}{3} & n_y \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly this matrix is symmetric. Thus, an element has been formulated with degrees of freedom $[\phi_1 \phi_2 U_1 W_1 U_2 W_2]$. Inserting such elements along the soil-fluid interface, the usual
solid and fluid elements can be used without any modifications. Detailed formulation can be found in the reference by Lotfi and Tassoulas [7].