4.2 OVERVIEW OF ELEMENT STIFFNESS MATRICES

In this section we outline the formulation of selected element stiffness matrices, with the intent of showing the conceptual simplicity of the process. Details of manipulations may be found in the text sections cited.

Bar. Figure 4.2-1 shows a straight bar whose nodal d.o.f. are axial displacements \( u_1 \) and \( u_2 \). A linear axial displacement field, as used in Section 3.9, is appropriate,

\[
\mathbf{u} = [\mathbf{N}] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{where} \quad [\mathbf{N}] = \begin{bmatrix} L - x & x \\ -1 & 1 \end{bmatrix}
\] (4.2-1)

Using Eq. 3.9-3, with \([\mathbf{B}] = \frac{d[\mathbf{N}]}{dx} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}/L\), we obtain

\[
[k] = \int_0^L [\mathbf{B}]^T AE [\mathbf{B}] \, dx = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\] (4.2-2)

where the latter expression is for a uniform bar, \( AE = \text{constant} \). The integral expression for \([k]\) does not demand that \( AE \) be independent of \( x \).

Beam. Figure 4.2-2 shows a four-d.o.f. straight beam element. Rotation \( \theta \) is assumed to be small, so that \( \theta \approx \frac{dw}{dx} \). Four d.o.f. define a cubic lateral displacement field,

\[
\mathbf{w} = [\mathbf{N}] \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}^T
\] (4.2-3)

where the four \( N_i \) are given in Fig. 3.13-2. The curvature field is \( w_{xxx} = [\mathbf{B}][\mathbf{d}] \), where

\[
[\mathbf{B}] = \frac{d^2}{dx^2} [\mathbf{N}] = \begin{bmatrix} -6 & 12x & -4 & 6x \\ L^2 & L^3 & L^2 & L^3 \\ -4 & 6x & 6 & -12x \\ L^2 & L^3 & L^2 & L^3 \end{bmatrix}
\] (4.2-4)

For constant \( EI \), the element stiffness matrix given by Eq. 4.1-14 is

---

**Figure 4.2-1.** Bar element with two d.o.f. (\( u_1 \) and \( u_2 \)).

**Figure 4.2-2.** Standard four-d.o.f. beam element.
\[ [k] = \int_0^L [B]^T EI [B] \, dx = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \] (4.2-5)

This \([k]\) operates on d.o.f. \(\{d\}\) having the order shown in Eq. 4.2-3. A slightly modified form of Eq. 4.2-5 is able to account for transverse shear deformation [4.2, 4.11].

The manipulations needed to obtain \([k]\) are shortened by using the "a-basis" of Eqs. 4.1-17 and 4.1-18. With \([X]\) and \([A]\) given by Eqs. 3.13-1 and 3.13-3, we write

\[ [B] = \frac{d^2}{dx^2} [X] [A]^{-1} = [0 \ 0 \ 2 \ 6x] [A]^{-1} \] (4.2-6)

\[ [k] = [A]^{-T} \int_0^L [0 \ 0 \ 2 \ 6x]^T EI [0 \ 0 \ 2 \ 6x] \, dx \ [A]^{-1} \] (4.2-7)

and obtain the same \([k]\) as given in Eq. 4.2-5.

The reader should understand the sign conventions for nodal moments and bending moment. Nodal moments \(M_1\) and \(M_2\) are positive when acting in the directions of \(\theta_1\) and \(\theta_2\) in Fig. 4.2-2. Bending moment \(M = EIw_{xx}\) is positive when it creates tension on the bottom of the beam. Therefore, \(M = -M_1\) at the left end and \(M = +M_2\) at the right end.

**Plane Frame.** A plane frame member can deform both axially and in bending. Effectively, to obtain a plane frame element we superpose the bar and beam elements of Figs. 4.2-1 and 4.2-2 (and of Eqs. 4.2-2 and 4.2-5). Nodal d.o.f. are \(\{d\} = [u_1 \ w_1 \ \theta_1 \ u_2 \ w_2 \ \theta_2]^T\). If the element is uniform and lies along the \(x\) axis, its stiffness matrix is

\[ [k] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \] (4.2-8)

If a frame element is arbitrarily oriented in the plane, its stiffness matrix is easily determined from \([k]\) of Eq. 4.2-8 by a coordinate transformation (see Section 7.5). Similarly, a plane truss element of arbitrary orientation can be obtained by coordinate transformation of Eq. 4.2-2 (again, see Section 7.5).

**Constant-Strain Triangle.** This element, shown in Fig. 4.2-3, is one of the earliest finite elements [1.8]. It can be used to solve problems of plane stress and plane strain. However, it is not a very good element for this purpose. We introduce it here primarily because it serves as a good example in subsequent discussions of why elements behave as they do.
Consistent Element Nodal Loads $\{r_e\}$

Beam Elements. Figure 4.3-6 shows nodal loads produced by typical loading patterns on a beam element. Nodal loads are calculated by use of Eqs. 4.1-6, 4.3-4, and 4.3-5. Specifically,

$$\{r_e\} = \int_0^L [N]^T q \, dx + [N]_{L/2}^T P + \left[\frac{dN}{dx}\right]_{L/2}^T M$$

(4.3-12)

Loads $\{r_e\}$ are of course work-equivalent to the original loads $q$, $P$, or $M$, in the sense defined in connection with Eq. 4.3-3. They are also statically equivalent; that is, loads $\{r_e\}$ produce the same resultant force and the same resultant moment about an arbitrary point as do the original loads, as the reader can easily show.

Error Produced by Lumping. The following example shows the merit of using consistent nodal loads rather than a lumping. Consider the uniformly loaded cantilever beam of Fig. 4.3-7a. Consistent nodal loads for a one-element model are shown in Fig. 4.3-7b. Taking $[k]$ from Eq. 4.2-5 and fixing the left end of the beam, we arrive at the following set of equations to be solved for $w_2$ and $\theta_2$,

$$\frac{EI}{L_T^3} \begin{bmatrix} 12 & -6L_T \\ -6L_T & 4L_T^2 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} qL_T/2 \\ -qL_T^2/12 \end{bmatrix}$$

(4.3-13)

Figure 4.3-7. (a) Uniformly loaded cantilever beam. (b) Consistent loads at node 2 of a one-element model. (c) Lumped (inconsistent) loads at node 2.
from which

\[ w_2 = \frac{qL^4}{8EI} \quad \text{and} \quad \theta_2 = \frac{qL^3}{6EI} \] (4.3-14)

These are the exact values of \( w_2 \) and \( \theta_2 \). (Values of \( w \) for \( 0 < x < L_T \) are not exact. The approximating field \( w = \Sigma N_i d_i \) is cubic in \( x \), but the exact field for a uniformly distributed load is quartic in \( x \).)

If the beam is divided into two or more elements of equal length \( L \), moment loads cancel at all interior nodes. Accordingly, lumped loading differs from consistent loading only in that lumped loading omits the clockwise moment \( qL^3/12 \) at the beam tip, thus causing tip deflection and tip rotation to be overestimated. With \( n \) the number of equal-length elements, lumped loading yields the following percentage errors in deflection and rotation at the right end.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Deflection error</th>
<th>Rotation error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>33.3%</td>
<td>50.0%</td>
</tr>
<tr>
<td>2</td>
<td>8.3%</td>
<td>12.5%</td>
</tr>
<tr>
<td>3</td>
<td>3.7%</td>
<td>5.6%</td>
</tr>
<tr>
<td>4</td>
<td>2.1%</td>
<td>3.1%</td>
</tr>
</tbody>
</table>

Consistent loading produces exact values of end deflection and end rotation for all values of \( n \).

4.4 EQUILIBRIUM AND COMPATIBILITY IN THE SOLUTION

In an exact solution, according to the theory of elasticity, every differential element of a continuum is in static equilibrium, and compatibility prevails everywhere. An approximate finite element solution does not fulfill these requirements in every sense. In the present section we note the extent to which equilibrium and compatibility conditions may be satisfied at nodes, across interelement boundaries, and within individual elements.

1. Equilibrium of nodal forces and moments is satisfied. The structural equations \( \{R\} - \{K\}\{D\} = \{0\} \) are nodal equilibrium equations. Therefore, the solution vector \( \{D\} \) is such that nodal forces and moments have a zero resultant at every node.

2. Compatibility prevails at nodes. Loosely speaking, elements connected to one another have the same displacements at the connection point. More precisely, elements are compatible at nodes to the extent of nodal d.o.f. they share. The latter statement allows the modeling of a physical hinge or roller between adjacent nodes that would otherwise be fully connected; one then connects some but not all nodal d.o.f. For example, if adjacent beam elements meet at a node where they share only translational d.o.f., a hinge connection is created.